

UNCLASSIFIED



Australian Government
Department of Defence
Defence Science and
Technology Organisation

Applications of Historical Analyses in Combat Modelling

Nigel Perry

Joint Operations Division
Defence Science and Technology Organisation

DSTO-TR-2643

ABSTRACT

While Lanchester's equations are commonly used as the basis for force-on-force combat models, it is important to remember that Lanchester's Equations are not a model of combat, only a model for combat attrition. There have been numerous attempts to compare historical combat data with the behaviour expected from Lanchester's Equations. The present work extends this comparison between historical battle data with behaviour expected from a battle where attrition is described by Lanchester's Equations. It examines how analyses of historical battles can contribute to the development of models of combat and hence our understanding of combat in addition to the processes used in the creation of databases of historical battle results. The historical data is compared against the expectations of both the deterministic and stochastic forms of Lanchester's Square Law.

RELEASE LIMITATION

Approved for public release

UNCLASSIFIED

UNCLASSIFIED

Published by

*Joint Operations Division
DSTO Defence Science and Technology Organisation
Fairbairn Business Park Department of Defence
Canberra ACT 2600 Australia*

*Telephone: (02) 6265 9111
Fax: (02) 6265 2741*

*© Commonwealth of Australia 2012
AR-015-190
December 2011*

APPROVED FOR PUBLIC RELEASE

UNCLASSIFIED

UNCLASSIFIED

Applications of Historical Analyses in Combat Modelling

Executive Summary

Lanchester's equations are commonly used as the basis for *force-on-force* combat models, even if only as a metamodel for a more complex combat model. It is important to remember that Lanchester's Equations are not a model of combat, only a model for combat attrition. The equations alone, therefore, cannot be expected to capture other effects such as the movement of engaged forces. There have been numerous attempts to compare historical combat data with the behaviour expected from combat models. To validate differential models of attrition, force and casualty numbers for both sides intermediate to the starting and finishing values are required. That level of detail is rarely available and often does not exist. An alternate approach using only the initial and final values of engaged force's strength has been tried previously. However, Lanchester's Equations describe the behaviour of a single system in time while the historical databases contain information about an ensemble of battles, each potentially with different values of attrition rate coefficients. The issue of why the results from such an ensemble follow the behaviour expected from Lanchester's Equations has never been adequately explored or explained.

The present work extends this comparison between historical battle data with behaviour expected from a battle where attrition is described by Lanchester's Equations. It examines how analyses of historical battles can contribute to the development of models of combat and hence our understanding of combat in addition to consideration of the processes that are used in the creation of databases of historical battle results. The implications of those processes on the limitations of this form of analysis, the constraints they impose and the resulting inherent biases are discussed, as well as methods that can be used to quantify and mitigate their effects.

The historical data is compared against the expectations of both the deterministic and stochastic forms of Lanchester's Square Law. However, it should be noted that examination of Lanchester's stochastic differential equation was not intended to be comprehensive or rigorous. Both have been covered extensively elsewhere, including by the author, and the present work contains numerous references to more authoritative works on these subjects for the interested reader.

Finally, evidence for considering battle as a particular type of complex adaptive system, one that involves co-evolution and scale free behaviours, is examined. It is proposed that this may be responsible for the unexpected observation that the behaviour of several parameters used to characterise combat is the same for both an ensemble of different battles and for the evolution of a single battle.

UNCLASSIFIED

UNCLASSIFIED

This page is intentionally blank

UNCLASSIFIED

UNCLASSIFIED

Author

Nigel Perry

Joint Operations Division

Dr Perry completed his B. Sc. (Hons.) at Melbourne University in 1980 and his Ph. D. at Monash University in 1985. From 1985 to 1989 he was a Research Fellow at Oxford University. From 1989 to 1991 he was a Research Fellow at Monash University. Between 1991 and 1996 he was a lecturer at Victoria University of Technology. He joined DSTO's Maritime Operations Division in 1996 and the Theatre Operations Branch in 1998. He is now a Senior Research Scientist in the Joint Operations Division.

UNCLASSIFIED

UNCLASSIFIED

This page is intentionally blank

UNCLASSIFIED

Contents

ACRONYMS, ABBREVIATIONS AND SYMBOLS

1. INTRODUCTION.....	1
1.1 Report Overview	2
2. MODELLING PARADIGMS AND THEIR APPLICATION IN COMBAT	
MODELLING.....	2
2.1 Reductionist	3
2.2 Holistic	5
3. THE ROLE OF HISTORICAL ANALYSES.....	6
3.1 Interpreting the Historical Record.....	7
3.2 Attrition Model Validation	8
3.3 Analysis for Ensembles of Battles	9
3.4 Issues in Database Development.....	11
3.5 Historical Database Instantiation	13
4. COMPARISON WITH PREVIOUS WORK	15
5. STOCHASTIC FORMS OF LANCHESTER'S EQUATIONS	22
5.1 Ito's Change of Variable Method.....	24
5.2 Further Application of the Change of Variable Method.....	25
6. ANALYSES OF THE DISTRIBUTION OF HISTORICAL DATA.....	26
6.1 Distribution of Initial Strengths	27
6.2 Distribution of Casualties	29
6.2.1 Segmented by Posture	30
6.2.2 Segmented by Outcome.....	31
6.2.3 Segmented by Force Size	32
6.3 Results Distributions in Helmbold's Relationship.....	33
6.3.1 Segmented by Battle Size.....	35
7. PREDICTIONS OF BATTLE OUTCOMES	36
7.1 Combat Entropy	38
7.2 Examination of the Corollary to Helmbold's Equation.....	39
8. BATTLE AS A COMPLEX ADAPTIVE SYSTEM.....	40
9. CONCLUSIONS.....	44
10. REFERENCES	46

**APPENDIX A: APPLICATION OF ITO’S RULE TO HELMBOLD’S
RELATIONSHIP..... 48**

Acronyms, Abbreviations and Symbols

a, b, c, d	constant attrition coefficients
$A(x, y), B(x, y)$	attrition/ drift coefficient functions
α, β	Equation of state parameters
σ_1, σ_2	constant variance coefficients
$S_1(x, y), S_2(x, y)$	variance rate functions
s	Arbitrary Exponents
$dx/dt, dy/dt$	Rate of change of force strength with time
$f(), g(), F()$	Arbitrary functions
$\partial f / \partial u$	partial derivative of f with respect to variable u
\ln	natural logarithm
$x(t), y(t)$	Strength of forces at time t
x_0, y_0	Initial force strengths
ρ	Correlation coefficient
X, Y	Ratio of forces current to original strength
V, μ	Defenders Advantage Parameters
$N(t)$	Arbitrary force strength
$c(t)$	Arbitrary force casualties
ν	frequency
\mathcal{S}	Entropy
$S(\nu)$	Frequency spectral density
R^2	Coefficient of determination
z_1, z_2	stochastic functions
dz	Wiener Process
ACW	American Civil War
LSR	Least Squares Regression
MLE	Maximum Likelihood Estimator
QJM	Quantified Judgement Model
$\langle \rangle$	Expectation value of the object within the brackets

This page is intentionally blank

1. Introduction

During the First World War F. W. Lanchester described one of the simplest, and most enduring, mathematical attrition models of force-on-force combat [1]. He proposed two systems of equations, depending on whether the fighting was “collective” or not. Collective combat between an attacking side of strength x and a defending side of strength y being described by the equations:

$$\begin{aligned}\frac{dx}{dt} &= -ay(t), & x(0) &= x_0 \\ \frac{dy}{dt} &= -bx(t), & y(0) &= y_0\end{aligned}\tag{1}$$

which result in the equation of state:

$$\frac{(x_0^2 - x^2)}{(y_0^2 - y^2)} = \frac{a}{b}\tag{2}$$

The quadratic form of which results in this system of equations being known as the Lanchester Square Law. Individual combat on the other hand is described by equations that produce an equation of state in which the force strengths are related linearly. The question of how each side’s strength is to be measured is deferred until the instantiation of a historical battle database is considered.

The assumption of “collective” combat is unlikely to apply throughout an entire battle and hence *real world* attrition results from a combination of collective and individual combats. This possibility has long been recognised and produced many attempts to generalise Lanchester’s system of equations to better represent actual combat results.

Lanchester’s model was developed as a description of air combat, in which each side was essentially composed of a single type of combat element. Force strength was then a simple matter of counting the number of aircraft in a side. Modern applications of Lanchester’s ideas to land combat run into the problem that each side consists of a number of types of combat element (infantry, artillery, tanks etc.) each of which interacts differently with each of the opposing sides’ combat types. The development of heterogeneous combat models is central to most current military combat simulations [2]. It is important to remember that Lanchester’s Equations are not a model of combat, only a model for combat attrition. The equations alone, therefore, cannot be expected to capture other effects such as the movement of engaged forces. This is frequently forgotten, as by Epstein [3].

There have been numerous attempts to compare historical combat data with the behaviour expected from Lanchester’s Equations, including the work of Helmbold [4] and Hartley [5]. Hartley also includes a comprehensive review of the effort to validate combat attrition laws using historical analysis. Recent work by the author has also investigated the ability of Lanchester’s Equations to describe patterns observed in the casualty statistics using Hartley’s

database of historical battles. This includes an examination of the inclusion of a fractal model of spatial dispersion on casualty values [6] and the distribution of casualties when Lanchester's Equations are modelled as stochastic processes [7].

1.1 Report Overview

The present work seeks to extend this comparison between historical battle data with behaviour expected from a battle where attrition is described by Lanchester's Equations. It begins with an examination of how analyses of historical battles can contribute to the development of models of combat and hence our understanding of combat. This is followed by consideration of the processes that are used in the creation of databases of historical battle results. The implications of those processes on the limitations of this form of analysis, the constraints they impose and the resulting inherent biases are discussed, as well as methods that can be used to quantify and mitigate their effects. This is followed by a brief review of how historical battle results have been and may be used to validate proposed attrition relationships, including examination of the presence of bias in the database using a sub-sampling approach.

Next, the author's previous work examining Lanchester's Equations modelled as stochastic processes is revisited and extended. However, it should be noted that examination of Lanchester's Equations and stochastic differential equation presented in the present work is not intended to be comprehensive, rigorous or complete. Both have been covered extensively elsewhere, including by the author, and the present work contains numerous references to more authoritative works on these subjects for the interested reader or reader unfamiliar with the use of Lanchester's Equations.

The author's previous analyses of Lanchester's Equations using historical battle data is then revisited, using the larger database developed for the present work. Finally, evidence for considering battle as a particular type of complex adaptive system, one that involves co-evolution and scale free behaviours, is examined. It is proposed that this may be responsible for the unexpected observation that the behaviour of several key parameters used to characterise combat is the same for both an ensemble of different battles and for the evolution of a single battle.

2. Modelling Paradigms and their Application in Combat Modelling

At its most basic, a model refers to a conceptual representation of some aspect of reality. Models are used when they are easier to understand than those aspects of reality they represent. Complex phenomena often require complex models if the model's behaviour is to reproduce that of the real world. However, while such models produce reasonable agreement with real world results, they are less often useful in understanding the functional dependence of the modelled quantity on the input parameters. In such cases it is useful to develop a

(simpler) model of that model which, although providing lower fidelity results, is better at explaining the causes of those results [2].

Models can be classified into three descriptive types [8], according to the degree of abstraction required:

- iconic models such as drawings or miniatures,
- analogue models in which other physical relationships represent the relationships under study, and
- symbolic models where abstract symbols or quantities are used to describe the real world.

Symbolic models are in turn divided into conceptual and mathematical models. Conceptual models include descriptions, plans and diagrams including charts. Mathematical models represent reality through quantitative relationships. They are further divided into static or dynamic depending on whether the model allows its variables to change over time or not. Mathematical models can also be classified as analytic or simulation depending on whether an exact closed form solution exists or whether a related sequence of models are used to converge to a solution for a complex problem. Simulation models are also divided into deterministic and stochastic models depending on whether uncertainty and risk is explicitly represented.

As usual a trade-off is involved when deciding which type of model to develop for a particular situation [9]. Simulation models capture real world complications better than analytic models. Analytic models can typically be solved under restricted conditions of population size or time, and are generally poor at describing transient behaviour. The most useful analytic models generally describe the real world using simple functional relationships.

There are two basic paradigms for developing mathematical models, including models of combat.

2.1 Reductionist

The reductionist approach attempts to describe a complex system by reducing the system to the interactions of its parts [10]. The component parts may also be reduced to simpler or more fundamental objects and the interactions between them. A complex system is viewed as no more than “the sum of its parts” in which all phenomena can be explained in terms of other, more fundamental, phenomena. Reductionism does not exclude the possibility of “emergent behaviour”¹, but does believe that such behaviour can be explained in terms of the phenomena from which they emerge. This paradigm is attractive to model developers as a complex model is obtained simply through the aggregation of simpler models of its fundamental behaviours.

¹ In the present work emergent behaviour is the way complex systems and patterns occur from many relatively simple interactions.

Taylor [11] has produced the most readily accessible and comprehensive reductionist treatise on force-on-force level Lanchester type models of attrition. It includes a theoretical treatment of differential equation models of attrition in force-on-force combat operations, providing both an introduction to and current overview of such models as well as a comprehensive and in-depth treatment of them. Both deterministic as well as stochastic models are considered. However, the resulting simplicity of a force-on-force level model limits their practical application.

Taylor goes on to identify which elements of a Lanchester type force-on-force level attrition model could be replaced with models composed of more fundamental objects in accord with the reductionist paradigm. These include attrition rate coefficients and force composition (homogenous versus heterogeneous), but does not explore how this might be achieved in practice.

The price of continuing the reductionist agenda and including models of component phenomena, is that the analytic approach to exploring the system properties has proven to have limited application. A simulation based approach is generally used instead.

The origins of quantitative models of combat attrition lie in the early 20th century among several authors working independently. Fowler [12] reviews the pioneering work by Chase, Fiske, Lanchester and Osipov, giving mathematical descriptions of their models. All of which can be regarded as variations of the same basic concepts and can be treated as particular cases of the generalised Lanchester Equations.

Fowler develops a detailed mathematical examination of the solutions to these equations using several different approaches, emphasising a number of particular combat types including both the linear and square law forms of the Lanchester equations. In addition to examination of Lanchester's coupled differential equations, Fowler includes an examination of the assumptions that underpin them and their solutions.

Fowler also attempts to combine historical analysis with the mathematical analysis of combat models. In common with other authors, such as Helmbold [4], he does not consider why the behaviour of data from a collection of unrelated battles can often be described using relationships developed from a model of the evolution of a single battle. Although the functional relationships between initial and final states are the same for such a collection of battles, their different and unrelated controlling parameters (such as attrition ratio) should have unconstrained values resulting in a range of final state values that obscure the functional relationships (as noted by Hartley [5]). Fowler's other comments of the use of historical analysis give the impression that he does not value its contribution highly, nor indeed any analysis that does not proceed from first principles. The work of Dupuy [13], while noted for its aesthetic form, is dismissed as merely empirical and lacking theoretical foundation. While a number of these criticisms are justified, the contribution that can be made by historical analysis and empirical studies in general are a subject of the present work.

In his pursuit of a description of combat attrition from first principles, most of Fowler's work consists of an examination of the derivation of values for the attrition coefficients that appear in Lanchester's Equations. Evaluation of these coefficients is one of the issues previously

identified by Taylor [11] as requiring further investigation. As part of this examination Fowler undertakes a comprehensive study of the wide range of factors that contribute to the evaluation of an attrition coefficient, including Bonder-Farrell theory of attrition rate coefficients, the theory of kill chains and heterogeneous engagements. The examination of stochastic behaviour of Lanchester's Equations, using the Fokker Planck equation, is less concerned with studying variability in combat outcomes than in justifying the standard equilibrium approximation whereby the variables in the equations are replaced by their expectation values. His findings are consistent with the application of the "law of large numbers", supporting a conclusion that Lanchester's Equations are more applicable for larger systems. The sensitivity of the system's evolution to its initial conditions is also explored. In addition to examining the equations governing the evolution of the expectation values for the system's variables, Fowler also developed equations governing the evolution of the variances and covariances of those variables. Noting the lack of correlation between force strength and combat success, he has advanced the idea that each side's perception of its chance for success is related to the behaviour of the system's variances and covariances.

The aggregation of one-to-one engagements into many on many engagements is extensively covered, again from first principles, starting with detection and tracking theory and considers time and range dependent effects on the probability of killing a target. The influence of weather, topology and weapon performance (ballistic or guided) is also included in his treatment of the determination of aggregate attrition coefficients, along with command and control problems in distributing targets among multiple firers.

Fowler's work is probably the most comprehensive review of combat attrition modelling presently available at the unclassified level and is a good guide to the complexity of large aggregate military models that have been used such as the USAF's Thunder and its replacements. However, the detailed scope of what is required by such a model also makes clear why such models fail to provide understanding. It is very difficult to relate trends in the model's outcome to causes, resulting in simpler, more empirical models such as Lanchester's Equations, retaining considerable utility.

2.2 Holistic

Phenomena such as emergence are believed to impose limits for the application of reductionism. In linear systems, the interactions between all components is obtained from the superposition of all possible pairwise component interactions. Nonlinearity may produce additional effects, not predicted by the properties of individual components or their simple interactions, in systems formed from large numbers of interacting components. Therefore at each stage in the aggregation of components to produce objects on a higher level of organisation, new concepts and generalisations must be added that do not arise from the properties of its components [14].

The holistic approach is the idea that all of the properties of a complex system cannot be explained by summing the behaviour of its component parts alone [15]. In contrast to the reductionist program above, aggregating a system's component parts is insufficient to provide a realistic description and requires additional phenomena or interactions which cannot be deduced from them in order to produce an accurate description. This model making

paradigm requires the developers to determine whether additional constructs are required by the model at each level of aggregation of its component parts. The reductionist viewpoint regards these additional phenomena as empirical and lacking rigorous justification.

Dupuy's Quantified Judgement Model (QJM) [13] typifies holistic combat models. It is an empirical expression of Clausewitz's "*Law of Numbers*", in which historical analysis of combat outcomes is used to determine approximate numerical values for its parameters. It is an example of the force scoring approaches reviewed by Jaiswal [16]. The combat power of a side is described in terms of its theoretical force strength and parameters describing the impact of operational factors. The force strength is a weighted sum of the lethality (killing power) of the elements making up that force. The weighting takes into account the impact of weather, terrain and the spatial dispersion of the force. Battle outcomes depend on the combat power of the two sides. The principal criticisms of the QJM approach are its complexity, often contradictory formulation, reliance on military judgement to determine values for certain parameters and its lack of a scientifically rigorous foundation [12]. However, this appears to be little more than the reductionist view of the holistic approach.

3. The Role of Historical Analyses

Both the reductionist and holistic approaches require data analysis. The main difference between them is that the reductionist approach only uses such analysis to determine the properties of the most basic, fundamental, objects in its hierarchical system deconstruction. These properties constitute the "first principles" from which all other behaviours can be developed. The data used at this level of analysis is generally at the scale of individual performance and interactions, which as a result does not include the effects of any collective behaviour.

In addition to making use of this data analysis of the basic objects, the holistic approach looks for additional properties and interactions that arise from collective behaviour at each level of object aggregation. This is obtained by additional data analysis of larger scale, collective, interactions including up to entire battles. Such data cannot in general be produced through controlled experiments or exercises and must be obtained from the historical record. This use of historical data analysis in the formulation of combat models is a major difference between these two approaches.

There have been numerous attempts to compare historical combat data with the behaviour expected from combat models, including the work of Helmbold [4] and Hartley [5]. Hartley also includes a comprehensive review of the effort to validate combat attrition laws using historical analysis. In contrast to those approaches, Hartley emphasises development of a combat model, including attrition, directly from historical battle data. Such analysis identifies relationships between many combat factors including force size, posture, casualties, surprise and duration. Mathematical expressions of these relationships are developed using standard regression analysis techniques and significance tests. Key casualty relationships are shown to be consistent with the expectations of the mixed Lanchester attrition equations.

To validate differential models of attrition, force and casualty numbers for both sides intermediate to the starting and finishing values are required. That level of detail is rarely available and often does not exist. Hartley's approach uses only initial and final values of engaged force strengths. A number of such compilations exist, which have been aggregated by Hartley. The component databases were created by different workers for a variety of different purposes. The potential for bias and error in the data is carefully considered, especially for battles prior to the 19th Century. Hartley argues that this database constitutes a random sample, because the individual datasets comprising the database were independently derived. While such aggregation will improve the accuracy of statistical estimators of dataset properties, the conclusion that while the database is not a true random sample it can be treated as if it were effectively random, requires further justification. It is difficult to avoid the conclusion that the database is little more than an aggregate of accidental sampling databases.

Lanchester's Equations describe the behaviour of a single system in time. However, the historical databases contain information about an ensemble of battles, each potentially with different values of attrition rate coefficients a and b . Should the results from such an ensemble follow the behaviour expected of a single system? Hartley has examined this issue at length. He considered several hypotheses, rejecting all save the conclusion that the relationship between the data from an ensemble of different battles was a direct consequence of the equations governing the attrition process. In other words, the behaviour governing an individual battle was reflected in the behaviour of an ensemble of battles. The bulk of Hartley's work is concerned with the development and examination of a model constructed from this historical analysis.

3.1 Interpreting the Historical Record

Despite the large number of recorded battles throughout history, the number with usable data is small. Any compilation of battle data, being a subset of all battles, constitutes a sample. A useful database will have a sample of battle data that is representative of patterns observed in the population of all battles. There are many issues which must be considered when attempting to use historical data, including:

- potential bias in narrative accounts of the battle due to most accounts being written by the victor or for propaganda purposes,
- many reported results are qualitative or approximate,
- many reported results for the same battle disagree, including dispute over which side won,
- when determining force strengths should support or service personnel be included,
- when determining casualties should prisoners be included,
- how should force strength be obtained from numbers of participants, should some form of force scoring such as the QJM be used,
- how should the effect of leadership, initiative, surprise, terrain and weather be included,
- how is the boundary of a battle defined, should strategic airpower or naval gunfire support be included,

- how should the effect of reserves be included, should the availability of uncommitted forces be included,
- should a battle be considered as a single event, or does it more closely resemble a related series of events (phases) separated in space and time.

This last point is critical. Most battles can be regarded as a series of events that occur both consecutively and concurrently. Both sides may be the attacker in different phases of the same battle, leading to dispute over who is the attacker. Each phase should strictly be considered as a separate battle, as originally intended by Lanchester [1]. However, under many conditions they can be aggregated into a single battle. How a large battle is segmented into smaller actions can substantially affect its analysis.

Most authors using historical data have attempted to address some of these issues [4, 5], especially questions of how to determine force strength and casualties. However there remains a question regarding the accuracy of much of the original reporting, especially for battles prior to the 19th Century. Because different methods and standards may have been used in recording each battle, the data will always contain inconsistencies and be subjective to some extent. But this is also true of current military activities and must be accepted as representing a limit on the accuracy of any analysis using real world data. These difficulties have left many in the field doubtful over the utility of historical analysis [12], as no universal solution exists to these problems. Accepting that such problems cannot be entirely eliminated, an approach to mitigate their impact will be considered in a following section.

3.2 Attrition Model Validation

As already mentioned, to validate differential models of attrition such as Lanchester's Equations, force and casualty numbers for both sides at times intermediate to the starting and finishing times are required. That level of detail is rarely available and often does not exist. The author is aware of four studies: Engel's [17] pioneering work on the Iwo Jima campaign, Busse's work on the Inchon campaign [18], Bracken's study of the Ardennes campaign [19] and Lucas's examination of the battle of Kursk [20]. As noted previously, Lanchester's Equations are not a model of combat, only a model for combat attrition. Each of these studies had first to segment the data, using narrative accounts of the battle, and extract those changes that were the result of attrition from all other changes. How a large battle is segmented into smaller actions can substantially affect its analysis. Each analysis made decisions regarding the inclusion of non-combat personnel, made more difficult by the combat effect of external participants such as US Naval gunfire at Iwo Jima and Inchon. The studies differed on the consistency of the historical data with the expectation from Lanchester's Equations. The Iwo Jima analysis found broad consistency, while the Kursk analysis found that the segmentation into phases was more important in explaining the observed casualty patterns. Much of the problem is due to the lack of sufficient data for both sides intermediate force and casualty values.

The four studies above each contained from 20 to 40 time correlated records of force strength and casualties for both sides. This sequence of values had to be segmented into smaller sequences due to the application of external variables such as the arrival of reinforcements. This resulted in sequences of related events available for analysis generally being much

smaller, and only rarely consisting of as many as 20 events. Determination of the form of the controlling attrition relationship is then made using regression analysis. However, studies of the relationship between sample size and precision in such analyses [21] show that 10 times as many observations are required as there are parameters in the regression model to obtain a 90 % confidence in the prediction of their values. Lanchester's Equations have 2 such parameters. The uncertainty arising from the use of the short data sequences available in the historical record is a major factor in the poor discrimination between competing attrition models reported in these studies.

3.3 Analysis for Ensembles of Battles

Lanchester's attrition equations describe the behaviour of a single system in time. However, as can be seen from Equation 2, the initial and final states of a battle are related. Hence a database of such information from a collection of historical battles does contain information about the attrition processes that govern their evolution. But, each battle in such compilations potentially has different values of the attrition rate coefficients a and b . Consequently, examination of the dependence of the left hand side of Equation 1 should not yield any interesting functional dependence, as the attrition ratio is independent of initial or final state values. This is not what is observed when the data is examined, as originally reported by Helmbold [4]. Hartley's data compilation [5] is shown in Figure 1 plotting the natural logarithm (\ln) of the left hand side of Equation 2 (known as the Helmbold Ratio) against the \ln of the initial force ratio.

The colour of each data point indicates the identity of the winning side. A more detailed examination of Hartley's data analysis follows later. While the data exhibits considerable scatter, a clear trend is apparent.

An early explanation for this behaviour was found if the attrition coefficients are not constants but depend on the force ratio. Such behaviour can be explained in terms of battlefield congestion preventing a side from making full use of its available forces and thus reducing the effective attrition rate against its opponent. Lanchester's Equations, modified to include the effect of "diminished marginal returns", are given in Equation 3:

$$\begin{aligned} \frac{dx}{dt} &= -f\left(\frac{x}{y}\right)y = -ax^s y^{1-s}, & x(0) &= x_0 \\ \frac{dy}{dt} &= -g\left(\frac{x}{y}\right)x = -bx^{1-s} y^s, & y(0) &= y_0 \end{aligned} \tag{3}$$

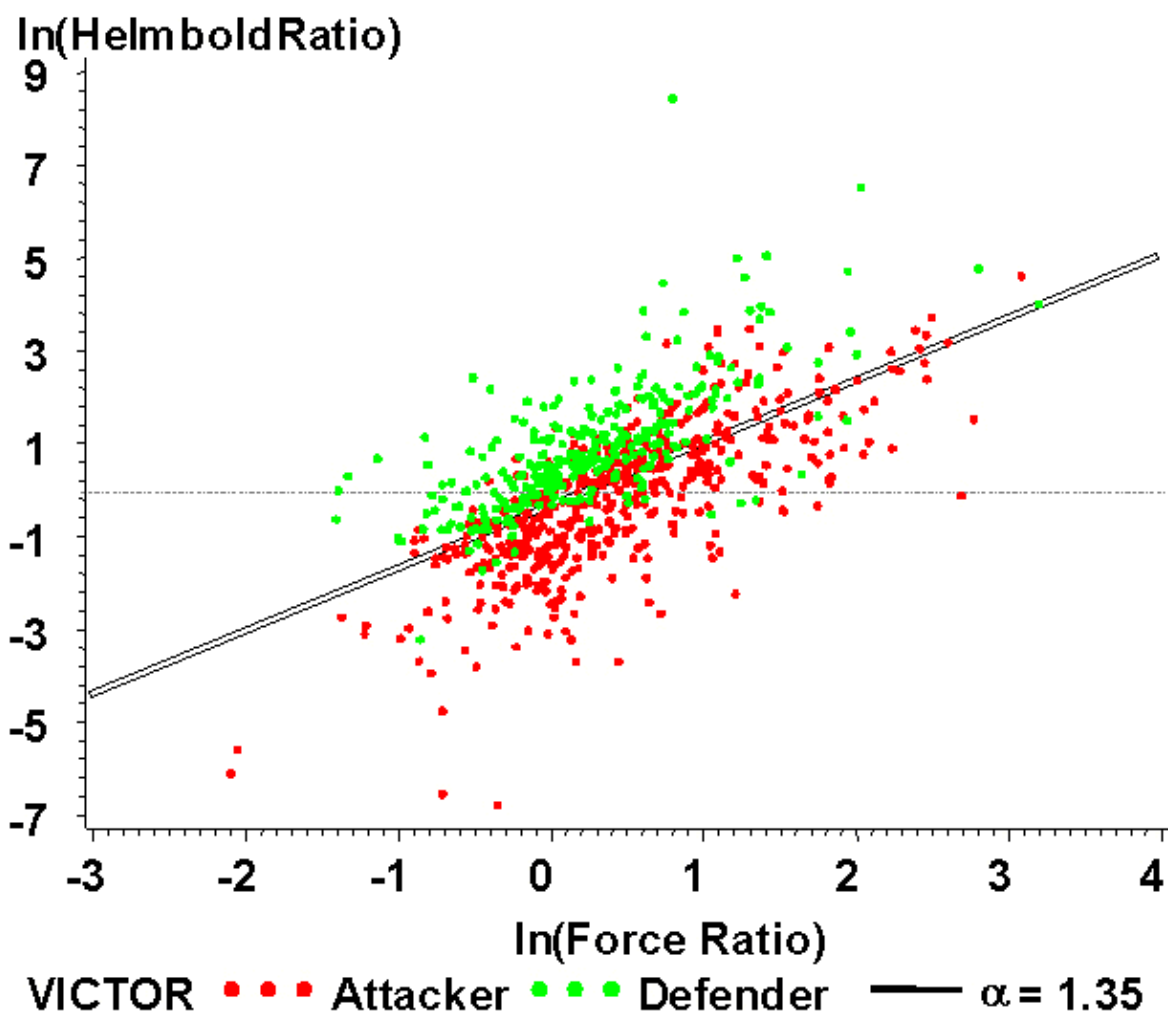


Figure 1: Helmbold's Relationship, from Hartley [5].

If the attrition functions $f(\cdot)$ and $g(\cdot)$ are simple power laws, as shown in the right hand expressions of Equation 3, the resulting equation of state (also known as the Helmbold Equation) [4] is shown in Equation 4. This equation is consistent with the equation of state in Equation 2; subject to the additional assumption above that "diminished marginal returns" constrain the values that the attrition coefficients a and b can take. This equation can also be regarded as a statement of that constraint.

$$\ln\left(\frac{x_0^2 - x^2}{y_0^2 - y^2}\right) = \alpha \ln\left(\frac{x_0}{y_0}\right) + \beta \quad (4)$$

Bearing in mind that equation 4 describes the evolution of a single battle, the similar behaviour shown by the data describing the ensemble of battles (Figure 1) was considered remarkable [5].

Previous work by the author [6] has shown that this equation of state also results from Lanchester's original equations, when the spatial distribution of each side's forces is modelled

stochastically where the probability is fractally distributed (power law). Both derivations are based on the same principle, a side's spatial distribution limits its ability to target enemy forces thus reducing its effective attrition coefficient which then depends on the force ratio.

Helmbold's pioneering work on historical battle analysis [4] made the assumption that the attrition coefficients were approximately the same for all battles. Hartley [5] sought to relax this assumption and has examined this issue at length. He considered several hypotheses rejecting all save the conclusion, albeit more empirical than rigorous, that the relationship between the data from an ensemble of different battles was a direct consequence of the form of the equations governing the attrition process of a single battle. In other words, the behaviour governing an individual battle was reflected in the behaviour of an ensemble of battles. Indeed, Helmbold's original work on the validation of Lanchester's Equations using historical data found this applies to a number of different parameters including the defender's advantage.

Further consideration as to why this similarity of behaviour exists will be deferred until Section 8. The present work will accept that the observed behaviour of such an ensemble of battles will follow the behaviour expected of that parameter during the course of a single battle.

3.4 Issues in Database Development

The fidelity of historical analysis is dependent on choice of an appropriate data sample. If the sample is representative of the population, parameter estimators derived from the sample will also be representative of the value of that parameter in the population, within error limits determined by sample selection and size [22]. How the sample is obtained has considerable influence on how representative of the population it is, with random sampling techniques considered the least influenced by bias and error.

A sample of objects from a population is random if all the members of the population have an equal chance of appearing in the sample. This applies to all members of the population, exceptional as well as typical members. Otherwise a correlation between the quantity being measured and probability of appearing in the sample can result in the value of the parameter's estimator being very different from the value of that parameter in the population.

When sampling a heterogeneous population the precision achieved can be increased and the risk of bias reduced by dividing the population into sections, each relatively homogeneous, and sampling each section (or stratum) separately. Estimates obtained for each stratum can then be combined to give the estimate for the whole population. If entire groups of a heterogeneous population are excluded from a sample, there are no adjustments that can produce representative estimates of the entire population. However, if some groups are under-represented and the degree of under representation can be quantified, then sample weights can compensate for the bias.

When the population being sampled is extensive or complex, the practical problems in taking a simple random sample are great, and the time taken for even a small sample may be large. The difficulty in obtaining a sample of a given size may be greatly reduced by carrying out the

sampling in two stages. First the complete population may be divided into a number of distinct primary units or sub-populations, and from these a sample is taken. From each of these sampled sub-populations a secondary sample, or sub-sample is taken.

The least useful and most subject to bias of all sampling procedures, accidental sampling, involves using what is available and most convenient as a sample pool.

Some of the difficulties in compiling historical battle samples have already been discussed above. A number of such data compilations were aggregated and used by Hartley. The component databases were put together by different workers for a variety of different purposes. The aggregate database covers a wide range of force ratios and while emphasising 20th Century battles, has reasonable coverage back to 1600. It emphasises land battles, but includes one air campaign. Hartley argued that the individual datasets comprising the database produce a random sample upon aggregation, because they were independently derived. This argument is similar to the inverse of Bootstrap sampling [23], which has been used to improve the accuracy of measures of sample statistical descriptors. Aggregation will improve the accuracy of statistical estimators, but does not affect bias. It is difficult to avoid the conclusion that Hartley's database is little more than an aggregate of accidental sample databases.

All battle databases are the product of the recursive application of the sub-sampling process. The population consists of all battles. This is first sampled to produce the set of all recorded battles. Many, especially smaller engagements, are never recorded. The requirement that both the initial and final values of forces strengths are known produces another sub-sampling stage to generate the set of all recorded battles with usable data. This sampling process also discriminates against smaller battles. Larger battles receive more attention and hence are more likely to have their attributes recorded. All battle databases are themselves samples of that sample. Even if the final sampling process was random, the process of recording history generates an intrinsic bias towards larger battles. This bias cannot be eliminated and any analysis technique must include procedures for identifying and dealing with that bias.

Hartley looked for the effects of bias in his data compilation by dividing it into a series of partitions using criteria such as date, size, attacker/ defender identity as well as campaign and original data source. Given that each of Hartley's data sources included a different spread of such values, each partition produced a different sub-sample of battles. Trends that were observed in the value of a sample estimator in a partition would indicate the presence of a correlation between the value of the parameter defining that partition and the probability of being included in the sample. That estimator would then be subject to bias. Hartley's analysis was primarily concerned with the behaviour of Equation 4 between the data partitions. He did not observe any significant differences, all observations being comparable to the estimated error, leading to the conclusion that the bias would not have measurable effects for his conclusions.

However, bias was observed by the author on examining the number of battles of particular sizes in the database [7]. How the way in which history is recorded leads to an inherent bias in such compilations has already been described. Each analysis technique must include procedures for identifying and dealing with that bias. Given this requirement for dealing with

bias, it is more important to establish consistency in the analysis from different data compilations. Consistency in the results from analysis of different databases, established using different methodologies and different primary sources, is an indication that bias in addition to the inherent historical bias above has little effect.

3.5 Historical Database Instantiation

Both Helmbold and Hartley sourced their data primarily from research undertaken in the first half of the Twentieth Century. Both compilations recorded significantly more information than just initial and final strengths. By restricting the data compilation undertaken for the present work to just the initial and final strengths increases the number of battles for which suitable data has been recorded. Moreover, in the last quarter of the Twentieth Century, significant amounts of new research has become readily available. A number of different factors has led to this increase, changed conditions in Europe has enabled researchers to access sources previously little explored, especially in Eastern Europe. Popular interest in military history has led to more detailed scrutiny of archives, enabling earlier work to be reviewed and long forgotten sources to be rediscovered. Although it is important to guard against revisionist tendencies among historians with a particular agenda to pursue².

The present work has developed a compilation of historical battle data, building on Hartley's compilation, using these advantages. This has enabled the number of battles included to increase from around 750 to around 1600. Each battle was checked against the most recent available data and earlier inaccuracies have been corrected. Previously, where some data was disputed, the battle had been included in the database multiple times with each entry corresponding to a different interpretation, such as when the winning side was disputed. More recent research has enabled most of these discrepancies to be resolved, and each battle now corresponds to a single entry. Where sources disagreed, the consensus opinion or values were followed. The internet has provided a means to facilitate large scale collaborative research on a scale not previously possible. Comprehensive archives, especially for subjects of popular interest such as the American Civil War [24] and Napoleonic Wars [25], have been produced by collaboration between enthusiasts and professionals using primary and secondary sources. Each entry in the database developed in the present work results from many sources and opinions³. This process is sometimes known as the "Wisdom of Crowds" [26] and is analogous to the process of Bootstrap sampling [23], with its resultant improvement in accuracy.

The availability of more sources, both primary and secondary, from both combatant and third party observers has enabled wider views on the progress and outcome of battles to be heard. While not preventing bias, which still occurs, the availability of alternate opinions mitigates against much possible bias in recent battle analyses.

The presence of service and support troops play an important part in the capacity of combat troops to engage in and sustain combat. Their contribution is sufficient to justify their

² Such as those seeking to restore Hans Delbruck's agenda or rehabilitate the reputation of Gen. D. Haig.

³ For an example of the detail now available for an increasing number of battles see: "1805: Austerlitz" by R. Goetz, Greenhill Books, London, 2005.

inclusion in battle strengths, where listed separately. While this has been done for service troops in direct support of a particular battle, service troops in a more general support role, possibly supporting several battles have not been included and in general their numbers are not known reliably anyway.

Where separately reported, prisoner numbers have not been included in casualty determination. When small, this represents at most a small error in the combat impact of losses. When large, the prisoners generally resulted from actions undertaken after the cessation of major combat and did not influence the outcome. This also applies to other non-combat casualties and is the reason for the exclusion of most sieges from consideration. The exception here is when the siege ended as the result of a single assault.

The most appropriate representation of a force's combat strength is to record the strength of each type of combat participant and develop separate attrition expressions for their interactions. Heterogeneous attrition models involve many interactions and the resulting combinatorial "explosion" greatly increases their complexity, which also reduces their utility and comprehension. Furthermore, historical data rarely includes detailed force compositions leading to considerable uncertainty in estimated values. A common way to reduce the attrition model's complexity is to construct a homogenous force strength determined using some form of force scoring methodology such as QJM. However, the effects of uncertainty in the composition of the force still remains. Indeed, given that a comprehensive historical database must include the effects of a wide range of weapons with considerable differences in lethality (comparing a spear with a modern Main Battle Tank for example), the likelihood exists that the resulting relative force strengths may be little more than an artefact of the force scoring methodology. It is not clear that such methods, at least for the purposes of the present work, are any more reliable than a simple comparison of the number of participants, which was the method chosen for this database development.

The decision as to what comprises a single battle, its boundary in space and time, is a decision that must be taken separately for each battle after considering the battle narrative. Each battle was selected for inclusion in an attempt to only consider battles that were thought to constitute a single engagement in terms of Lanchester's original conception. The timing and availability of reserves also affected this decision as well as the force size.

As mentioned previously, the bias introduced by the process of recording history is intrinsic and must be allowed for in subsequent analysis. If no suitable reason for choosing one source over another existed, the author made a judgement call on which version would be used. While this is unlikely to be always correct, it is at least self-consistent in the presentation of data.

The data recorded for each battle included its identifying name and year, as well as a generic identifier describing the conflict and technology/tactics employed to facilitate segmentation of the database into groups of roughly similar battles. For both sides of the battle the initial and final strengths are recorded as well as that side's principal posture (attacker, defender) and its final status (winner, loser). A summary of which appears in the following table.

Table 1: Dataset Segmentation and Summary

Data-segment Epoch	Start Year	End Year	Number of Battles	Attacker Victories	Defender Victories
Ancient	-490	1598	63	36	27
17th Century	1600	1692	93	67	26
18th Century	1700	1798	147	100	47
Revolution	1792	1800	238	168	70
Empire	1805	1815	327	203	124
ACW	1861	1865	143	75	68
19th Century	1803	1905	126	81	45
WWI	1914	1918	129	83	46
WWII	1920	1945	233	165	68
Korea	1950	1950	20	20	0
Post WWII	1950	2008	118	86	32

4. Comparison with Previous Work

The population of all battles throughout history cannot be documented. Recorded history is only a sample of those events that took place, and as already described, that sample is fundamentally biased and accidental in nature. Random sub-sampling of those recorded events does not change this basic property of the resulting sample. However, if different methodologies for sub-sampling produce consistent results then the observed patterns of behaviour can be considered as indicative of behaviour in the source sample and not artefacts of the sub-sampling process. In particular, if analysis of the data gives the same results, both before and after the effects of bias in the data has been addressed, the result can be considered as insensitive to the effect of bias and indicative of actual behaviour in recorded history. Comparison of the behaviour of the database developed in the present work with Hartley's database provides this consistency check.

Not all of Hartley's approaches to segmenting his database have been examined here. Attacker/ Defender pairs were not examined as they contain too few data points to provide worthwhile analysis. The effect of outliers was not considered. This study examines the distributions of many results around their mean values. Whether a datum is an outlier or an instance of an extreme (low probability) event will have a strong effect on the tail of the resulting distribution. Bias could be introduced by arbitrarily removing data points from consideration based on perceived differences in behaviour, when such data points may be useful in illustrating dependencies.

Equation 4 describes the key relationship to be used for comparison of Hartley's work, Figure 1, with the database developed in the present work. The analogous results for the current database are shown in Figure 2.

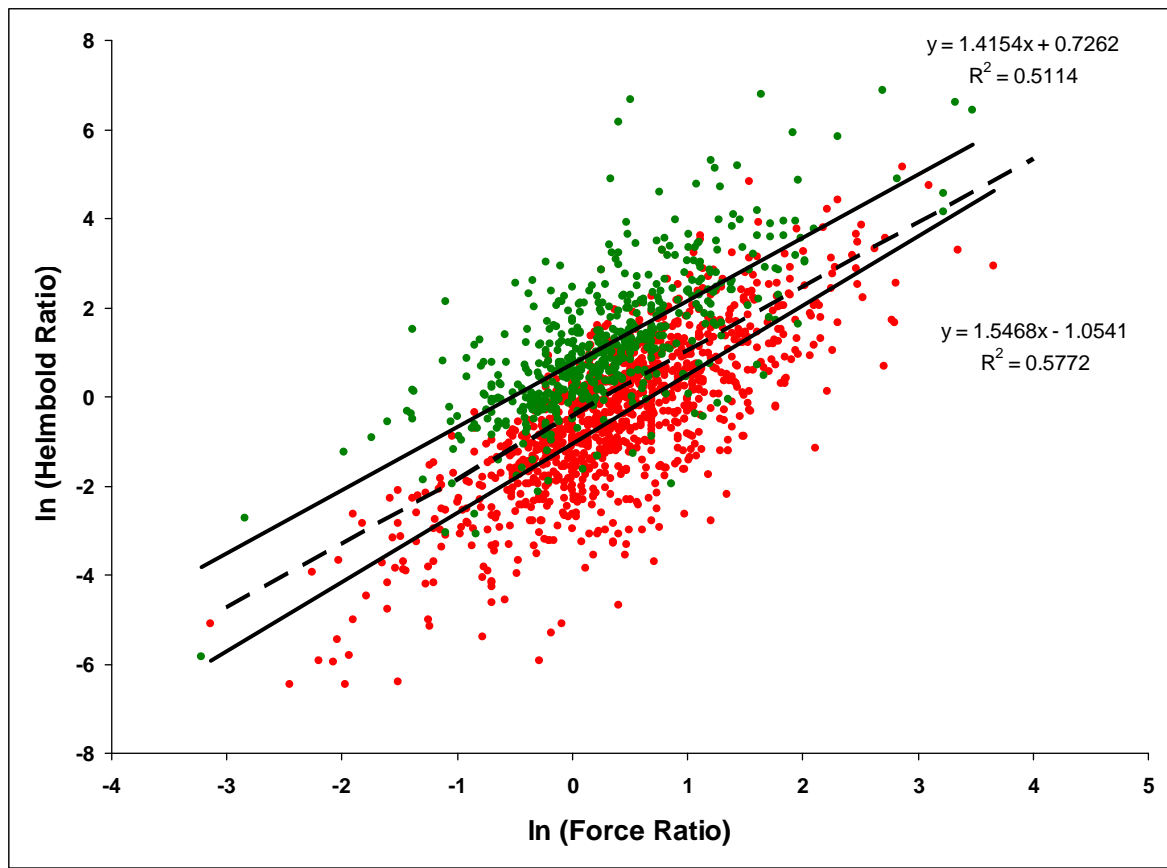


Figure 2: *Helmbold's Relationship using the current database. Attacker victories are coloured red while defender victories are green.*

Least squares regression generated the best fit lines shown. The dashed line represents the full data set with solid lines representing the data segmented by the victor's posture. Equations describing the best fit lines for attacker victories and defender victories are also shown. Attacker victories are common for data both above and below the overall best fit line (dashed) describing the average battle outcome, while defender victories occur predominantly above that line. Attackers initially hold the initiative in battle, in that they can choose whether to attack or not. If their assessment of the likelihood of success is not favourable they will generally choose not to attack, which may explain the difference between attacker and defender successes shown in Figure 2.

The gradient α in Equation 4 is the principal parameter characterising the behaviour of the datasets. It determines how sensitive a battle's outcome (specified by the Helmbold Ratio) is to changes in the initial force ratio. If α has a dependence on force size (not ratio), technology (spears versus machine guns), winner's posture (attacker/defender) or another significant discriminating factor between battles, then analysis of this dataset would be subject to bias. If α does not exhibit such behaviour, the results drawn from such analysis can be considered as insensitive to such effects.

The regression parameters obtained for the dataset as a whole, and for the dataset segmented according to the victor's posture as well as the battle's epoch are given in Table 2 below. The

value for α , the standard error in α , the regression coefficient of determination and the maximum and minimum values for α using a 95% confidence interval are given for each dataset. Values for Hartley's database are also given for comparison. The Korea epoch contained too few entries to undertake this analysis and was not considered.

Table 2: Dataset Segmentation Regression Parameters

Dataset	α	err	R ²	min α	max α
Unsegmented					
Hartley's Database	1.38	0.06	0.41	1.26	1.50
Current Database	1.44	0.04	0.43	1.36	1.52
Segmented by Posture					
Attacker Victories	1.55	0.04	0.58	1.46	1.62
Defender Victories	1.42	0.06	0.51	1.30	1.53
Segmented by Epoch					
Ancient	2.48	0.34	0.49	1.80	3.16
17 th Century	1.62	0.34	0.20	0.94	2.30
18 th Century	1.47	0.12	0.50	1.23	1.72
Revolution	1.30	0.11	0.36	1.07	1.53
Empire	1.27	0.10	0.32	1.07	1.47
ACW	1.02	0.15	0.25	0.72	1.31
19 th Century	1.82	0.12	0.66	1.59	2.06
WWI	1.22	0.11	0.48	1.00	1.44
WWII	1.20	0.11	0.35	0.99	1.42
Post WWII	1.63	0.10	0.67	1.41	1.84

The current database values for maximum and minimum α , as well as those segmented by posture, define an overlap in the region 1.46 to 1.52. This common region is also consistent with Hartley's results. More variation in the value of α is observed between each epoch. If, for the moment, the values from the Ancient and American Civil War (ACW) epochs are ignored as outliers, examination of the values for maximum and minimum α again show a good overlap, although not as good as when segmented by posture.

The agreement in the values of α observed in Table 2 is as good as that found by Hartley. Within a 95% confidence interval the possibility that a single value for α characterises each of these datasets cannot be discounted. The observed values of α may then be regarded as indicative of its value in the overall population and not an artefact of the sampling process.

The low values for the coefficient of determination are also significant. One standard interpretation of this value is the fraction of the observed variation in the natural logarithm (\ln) of the Helmbold Ratio that can be explained by the variation in the value of the \ln of the Force Ratio. The small values indicate that other factors are responsible for most of the observed variation. A possible interpretation of this variation will be explored in a later section.

Explanation of the value for α observed for the Ancient epoch requires closer examination. The value of the Helmbold Ratio is shown plotted against its corresponding Force Ratio in

Figure 3, segmented by the victor's posture. Least squares regression generated the best fit lines shown. The dashed line represents the full epoch data with solid lines representing the data segmented by the victor's posture. The observed value of α for attacker victories is similar to that observed for defender victories. Both values of which are significantly lower than the 2.48 reported for the epoch as a whole. Against expectation, in this dataset attacker victories are more common for lower Force Ratio values while defender victories are more common at higher Force Ratio values. Regression of the dataset as a whole has correlated these low Force Ratio attacker wins with the high Force Ratio defender wins, resulting in the large observed value of α . This effect occurs to some degree in all epoch data segments, but is more pronounced here due to the large vertical separation between the attacker and defender sub-sets. A similar conclusion can be drawn for the ACW epoch.

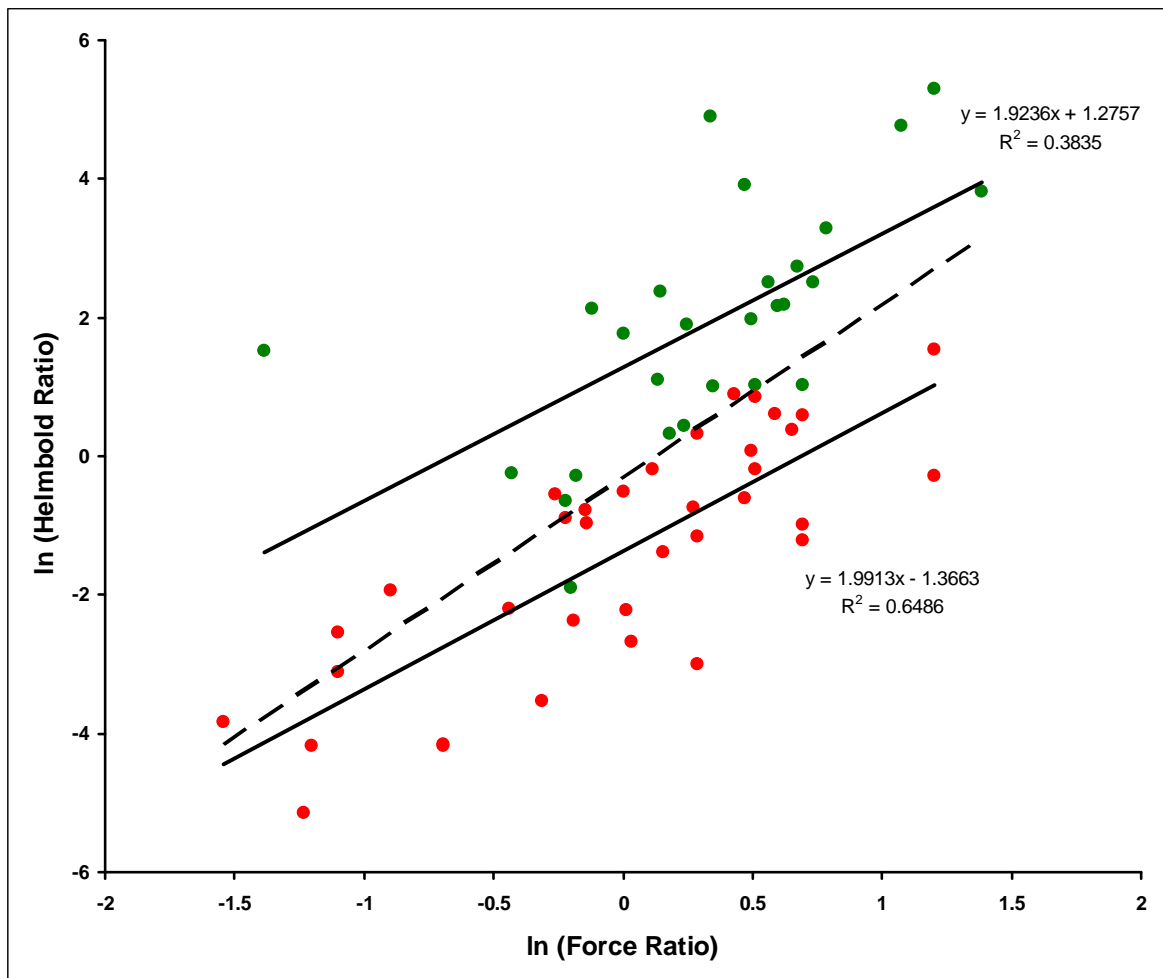


Figure 3: *Helmbold's Relationship for Ancient Epoch battles. Attacker victories are coloured red while defender victories are green.*

In most of the analyses in the present work, attacker and defender values will be separated to prevent this form of aliasing biasing the results.

It is important to determine whether any systematic trends in the value of α exist. A trend over time (and hence possibly of technology) can be examined using the data from Table 1

above. To enable comparison with Hartley's examination of this trend, the data was not segmented by the winner's posture. Ignoring the Ancient epoch (outlier), a year representative of each epoch was found by determining the average year for all battles constituting the epoch. The value of observed α plotted against its representative year is shown in Figure 4.

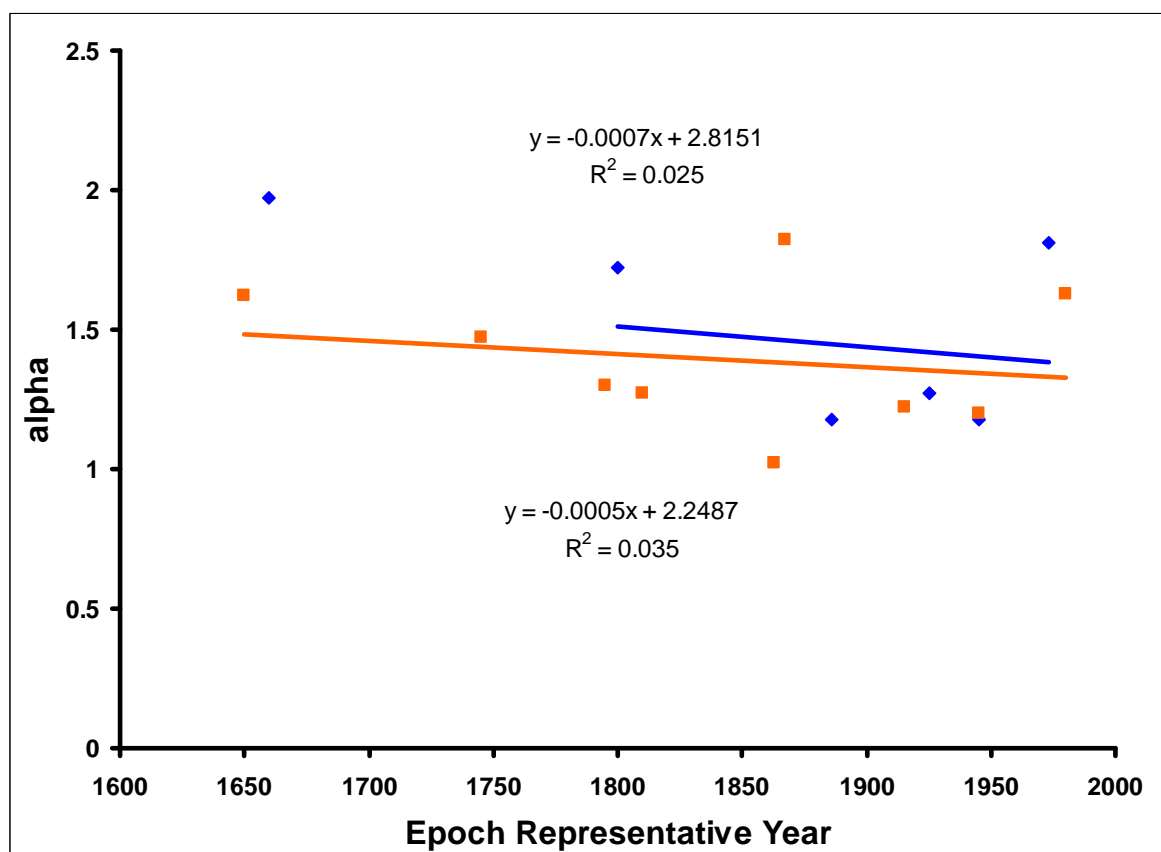


Figure 4: The value of the gradient α against representative year. Current database values are coloured orange and Hartley's values are coloured blue.

It can be immediately noted that Hartley's values are consistent with the data from Table 2. The small values for the coefficient of determination indicates that the systematic changes in α over time are not significantly different from zero. More importantly, this means that changes related to time (such as technology) do not have a significant impact on the outcome of battles.

A trend in α with battle size would also be significant. It is a little more difficult to quantify as size can be determined in a number of ways. Most workers define the size of a battle as the total of all forces involved in the battle (both sides). This can be misleading as Helmbold [4] reported that the attacker's strength is not strongly correlated with the defender's strength. For small and mid-sized battles (up to 40000 per side), the correlation between the strengths of the two sides is poor, as can be seen in Figure 5. Using the total strength as a measure of size can then mask trends that depend on a side's strength.

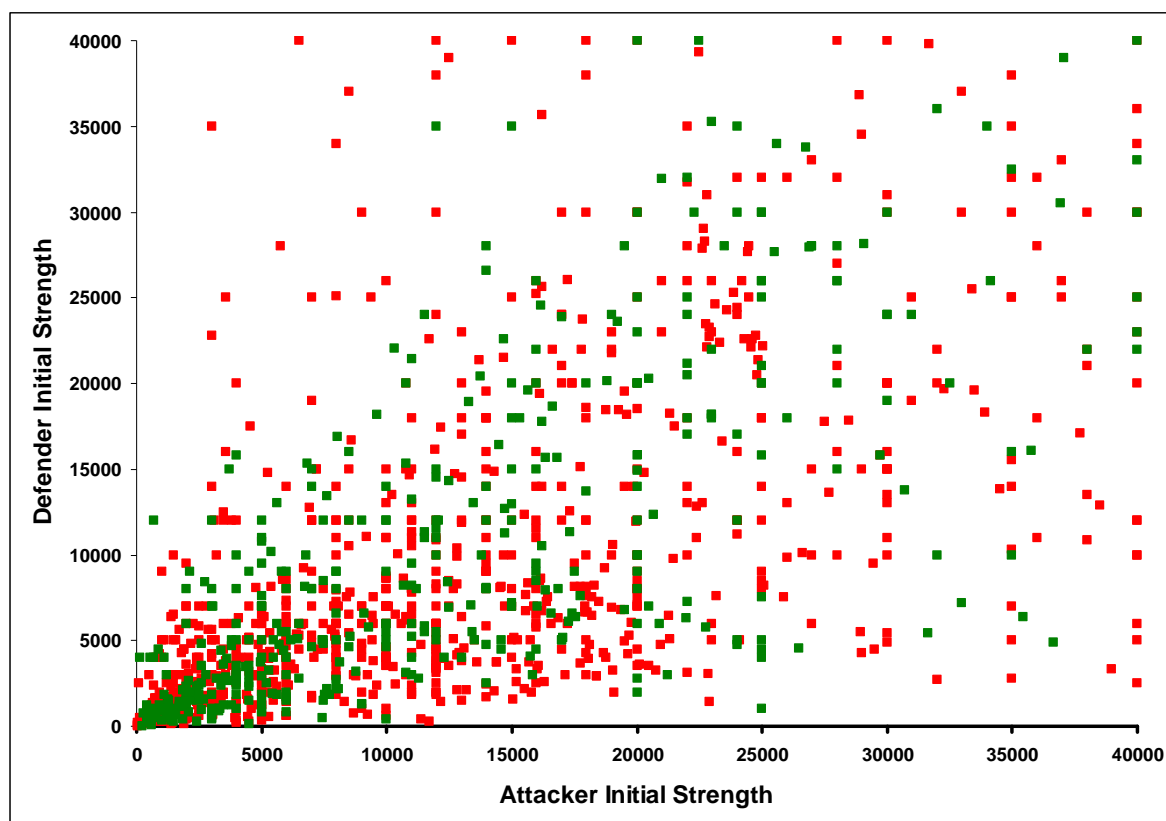


Figure 5: The Defender's initial strength as a function of the Attacker's initial strength. Attacker victories are coloured red while defender victories are green.

Two different means of quantifying battle size were examined: the size of the attacking force and the size of the defending force. Dividing the 1083 battles for which the attacker was victorious into quartiles using these factors enables a comparison of the value of α , determined from regression analysis in each quartile, for different sized battles. The results are given in Table 3.

Table 3: Measurement of Attacker Victories' α by battle size Quartiles

Quartile	Data Ordered By: Defender's Strength		Attacker's Strength	
	α	err	α	err
1	1.53	0.08	1.56	0.08
2	1.60	0.09	1.50	0.08
3	1.73	0.09	1.24	0.09
4	1.91	0.07	1.59	0.07

Clearly α does not depend on battle size when determined by the attacker's force size, but does depend on battle size when determined by defender's force size. This complex dependency on battle size was not observed by Hartley, who classified battles using total strengths involved.

Given the unequal range of battle sizes in each quartile, it is informative to consider the dependence of α on actual battle size. The average battle size for each quartile was taken as representative of that quartile. These results are shown in Figure 6.

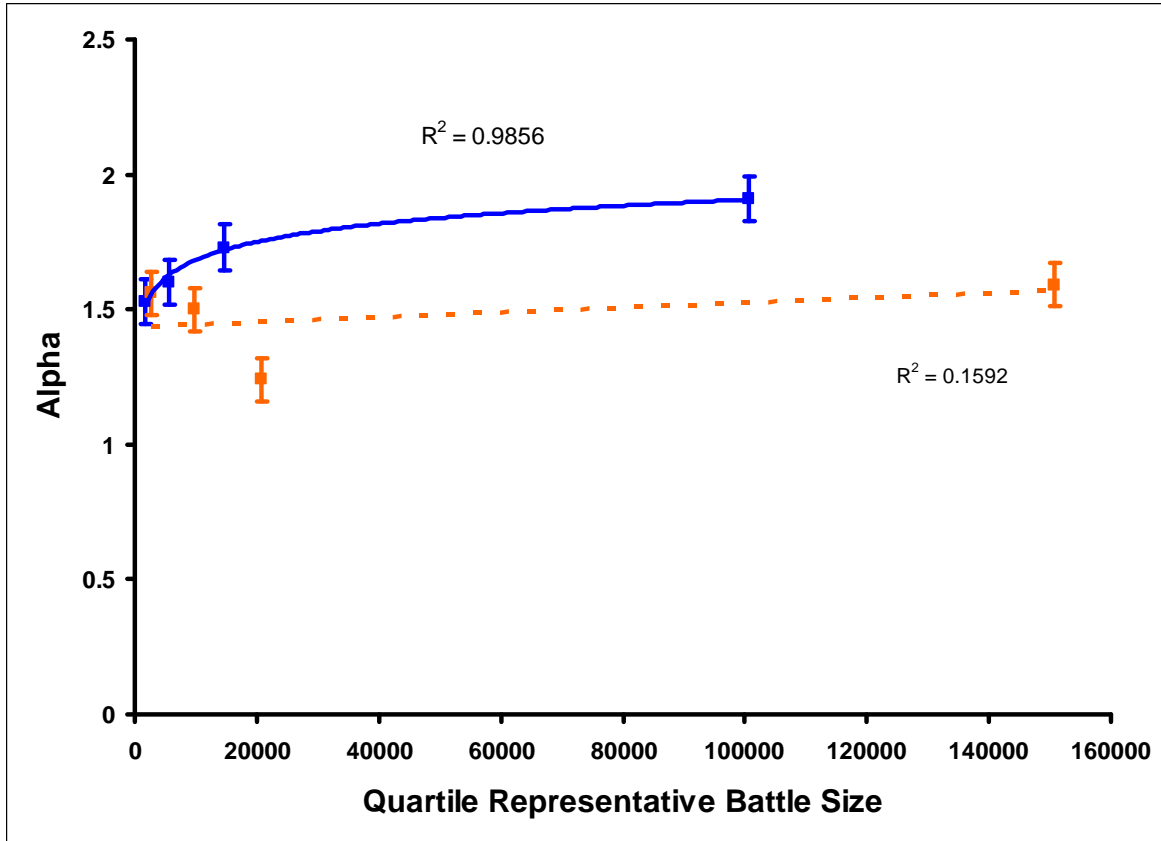


Figure 6: The value of the gradient α against Battle Size. Battle sizes determined by Attacker size are coloured orange and by Defender size are coloured blue. Standard error bar sizes are shown for all points.

A logarithmic dependence for α on defender's size is observed. Logarithmic dependences on battle size have been observed in many parameters related to combat [4]. A possible explanation for the observed behaviour can be found on further consideration of the Lanchester's Equations used in this analysis, Equation 3, in which the rates of change are determined by a single term. If however, the attrition rates are governed by a "mixed law" Lanchester Equation [12] with a polynomial strength dependence such as:

$$\begin{aligned} \frac{dx}{dt} &= -ay + cy^2 + \dots \\ \frac{dy}{dt} &= -bx + dx^2 + \dots \end{aligned} \quad (5)$$

the residual behaviour of the non-linear terms could result in an apparent dependence of the coefficient α on force strength as observed. The effect of such terms have been considered by

Woodcock and Dockery [27] in their examination of the use of Catastrophe Theory for combat modelling.

Catastrophe Theory is a method for examining non-linear dynamics where the potential function describing the system's evolution is treated as a folded manifold [28]. Such manifolds allow multiple input values, for some parameter, for the same output value over a restricted domain. The potential function defining the manifold is obtained from the dynamics of the system under study. Catastrophe Theory analyses degenerate critical points of the potential function where not just the first derivative, but one or more higher derivatives, of the potential function are also zero. These are called the *germs* of the catastrophe geometries. The degeneracy of these critical points can be examined by expanding the potential function as a power series in small perturbations of its parameters. There are nine basic catastrophe types.

The major problem for the application of Catastrophe Theory to combat is in the definition of the manifold potential. Current theories of combat are only able to produce part of the germ for any of the basic catastrophe types. Unfortunately, the terms in the germ that had no mechanism to support their inclusion are critical for the catastrophe behaviour. Catastrophe Theory was not considered further in the present work.

This completes the comparison of results using the database developed for the present work with previous results. It should be clear that the results presented above are consistent with Hartley's findings. The lack of variation observed in the value of the coefficient α , using different methods to segment the database, supports the conclusion that the bias inherent in historical data does not produce an observable effect in the analyses. Therefore, the database may be taken as representative of the real world and not merely as an artefact of the sampling methodology.

5. Stochastic Forms of Lanchester's Equations

Put simply, a stochastic process is described by a variable whose value changes in time in an unpredictable way. Such processes can be discrete, when the variable's value can only change at specified fixed points, or continuous when the value can change at any time. Stochastic processes may also take continuous values, when the underlying variable can take any value within a specified range, or discrete values where only certain specified values are allowed.

A Markov process is a particular type of stochastic process where only the current value of a variable is relevant for predicting its future evolution. A continuous time, discrete value Markov process has been demonstrated to produce a stochastic attrition model analogous to Lanchester's deterministic equations [29]. Most modern combat simulations use Markov processes to describe attrition. The stochastic theory of attrition has been comprehensively explored by a number of workers and is readily accessible [29].

Stochastic analogues of Lanchester's Equations are a specific case of the stochastic linear system of equations for two stochastic variables:

$$\begin{aligned} dx &= A(x, y)dt + S_1(x, y)dz_1, & x(0) &= x_0 \\ dy &= B(x, y)dt + S_2(x, y)dz_2, & y(0) &= y_0 \end{aligned} \quad (6)$$

where A and B are functions of the stochastic variables x and y , and possibly also of time t , that describe the regular and stochastic evolution of x and y . The functions S_1 and S_2 describe the magnitude of the stochastic variance in x and y resulting from the action of the stochastic functions z_1 and z_2 . The form of z_1 and z_2 depends on the type of stochastic process being investigated. Normal probability distributions are generally used for continuous stochastic variables where the law of large numbers is assumed to hold. Although continuous variables are only an approximation, the difference between adjacent allowed values of the force strength variables is much smaller than the magnitude of the strengths themselves. As such, this will be the only case considered in the present work.

Following the approach of Black and Scholes [32], stochastic analogues of Lanchester's Square Law are obtained when linear dependences on the complementary variables are used in Equation 6. This substitution produces Equation 7, which is the same as the system studied by Amacher and Mandallaz (their Equation 6):

$$\begin{aligned} dx &= -(adt - \sigma_1 dz_2)y, & x(0) &= x_0 \\ dy &= -(bdt - \sigma_2 dz_1)x, & y(0) &= y_0 \end{aligned} \quad (7)$$

Most studies of the application of stochastic forms of Lanchester's Equations have tended to emphasise its Markov process properties [11, 12]⁴. The present work will examine the relationship between observed behaviours and the parameters defining this stochastic system separately. It is not proposed to provide comprehensive or rigorous coverage of the methods of stochastic calculus or their application to the study of analogues to Lanchester's Equations. A rigorous treatment of stochastic calculus theory can be found in the book by Klebaner [30] and its specific application to solution of attrition equations in the work of Amacher and Mandallaz [31]. Using a matrix formulation of the system, they developed an analytic general solution for the expectation values of the stochastic variables (actually the square of the stochastic variables) in the form of an infinite series product of matrices. Interestingly, they chose not to report any further analytic investigations into the behaviour of the system, preferring instead to examine their solutions numerically with an examination of synthetic data obtained through simulation of the behaviour of the dynamical system. The same starting conditions were used and the results of 1000 runs of the system were studied. Their conclusions are interesting. Battle duration had a highly skew distribution, to the extent that mean battle duration was found not to be a useful summary statistic. Significant departures from a normal distribution were observed for casualty values. They considered this surprising, which is in itself surprising since the emergence of a log-normal distribution for casualties is a clear consequence of the model. The outcomes of the simulations were found to be sensitive to small variations in the initial conditions. A key conclusion of their work was the importance of battle termination conditions for the simulation's outcome. In short, the

⁴ By which is meant an examination of the behaviour resulting from the stochastic nature of Equation 7 without separating the effects resulting from the mean and variance terms inside the brackets of Equation 7.

general solution while interesting, is not particularly useful. This pioneering work does not appear to have been followed up.

Previous work by the author [7] also examined solutions to stochastic forms of Lanchester's Equations (Equation 7). In contrast to Amacher and Mandallaz, a general solution was not sought, instead focussing on solution of approximations to the equations themselves. This enabled the functional form of the force strength variables to be more easily identified and compared with historical data using the same approach as that outlined in the present work. A log-normal distribution for casualty values was found which was consistent with the expectation of the approximated equations and that observed by Amacher and Mandallaz [31]. A further difference between Amacher and Mandallaz and the author's work is in the interpretation given to the origin of the stochastic term in Equation 7, which they regarded as merely the result of time dependent random disturbances of the attrition processes. The author's previous work has attempted an interpretation of this contribution and shown how it can arise from interactions between the "system" of forces in combat and the remaining non-combat processes affecting those forces.

5.1 Ito's Change of Variable Method

A standard approach in stochastic calculus is to employ a change of variable method, commonly ascribed to Ito, to explore the evolution of quantities defined from the system's stochastic variables [30]. This approach is essentially the same as that used by Fowler [12] who used the Fokker-Planck equation to examine the transition probability density in his formulation of stochastic attrition. This approach has also been used to derive the Black-Scholes differential equation [32].

Let $f = f(x, y, t)$ be a function of two stochastic variables and time such that f is twice differentiable in x and y and also once differentiable in t . Following Navin's [30] informal approach to the development of Ito's lemma:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{1}{2} \left(\frac{\partial^2 f}{\partial x^2} \langle dx dx \rangle + \frac{\partial^2 f}{\partial y^2} \langle dy dy \rangle + 2 \langle dx dy \rangle \frac{\partial^2 f}{\partial x \partial y} \right) + \dots \quad (8)$$

This expression can be simplified, if the relationships between the dynamical variables are known, by substituting the first and second order differentials into Equation 8. This rule can be applied to the examination of stochastic forms of Lanchester's Equations by first limiting consideration to systems that are described by Equation 6. This provides the first order differential terms. Using simplified notation [30] (ignoring the delta functions), on taking the expectation values, the second order terms are similarly obtained. For example:

$$\langle dx dx \rangle = \langle (Adt + S_1 dz_1)(Adt + S_1 dz_1) \rangle \rightarrow A^2 dt^2 + 2S_1 dz_1 dt + S_1^2 dz_1^2 \quad (9)$$

However, as $dt \rightarrow 0$, it can be shown [30] that:

$$\begin{aligned}
\langle dz^2 \rangle &\rightarrow dt \\
\langle dt^2 \rangle &\rightarrow 0 \\
\langle dtdz \rangle &\rightarrow 0
\end{aligned} \tag{10}$$

which gives:

$$\begin{aligned}
\langle dx^2 \rangle &\rightarrow S_1^2 dt \\
\langle dy^2 \rangle &\rightarrow S_2^2 dt
\end{aligned} \tag{11}$$

Both the z_1 and z_2 terms from Equation 6 can then be replaced by dt , as the integrations over z_1 and z_2 are independent *except for the cross term*, from which follows:

$$\langle dxdy \rangle \rightarrow S_1 S_2 dz_1 dz_2 \tag{12}$$

Which in the integral over the stochastic processes can be replaced by

$$\langle dxdy \rangle = S_1 S_2 \rho dt \tag{13}$$

Where ρ is the correlation coefficient between the two stochastic processes which have an as yet unspecified degree of coherence. This allows Ito's rule, Equation 8, to be written as:

$$\begin{aligned}
df = & \left(\frac{\partial f}{\partial t} + A \frac{\partial f}{\partial x} dx + B \frac{\partial f}{\partial y} dy + \frac{1}{2} S_1^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} S_2^2 \frac{\partial^2 f}{\partial y^2} + S_1 S_2 \rho \frac{\partial^2 f}{\partial x \partial y} \right) dt + \\
& \left(S_1 \frac{\partial f}{\partial x} + S_2 \rho \frac{\partial f}{\partial y} \right) dz
\end{aligned} \tag{14}$$

for dynamical systems described by Equation 6. This expression can then be used to examine how any function f defined for the system under study changes over time.

5.2 Further Application of the Change of Variable Method

The relevant Ito change of variable rule is further simplified by explicit specification of the undefined functions of Equation 14. When the instantiation of the general linear system that is given in Equation 7 is considered, the change of variable rule applicable to the stochastic analogue of Lanchester's Square Law is obtained.

$$\begin{aligned}
df = & \left(\frac{\partial f}{\partial t} - ay \frac{\partial f}{\partial x} dx - bx \frac{\partial f}{\partial y} dy + \frac{1}{2} \sigma_1^2 y^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} \sigma_2^2 x^2 \frac{\partial^2 f}{\partial y^2} + \sigma_1 \sigma_2 \rho xy \frac{\partial^2 f}{\partial x \partial y} \right) dt + \\
& \left(\sigma_1 y \frac{\partial f}{\partial x} + \sigma_2 \rho x \frac{\partial f}{\partial y} \right) dz
\end{aligned} \tag{15}$$

A number of quantities of interest for the Lanchester system with constant rate coefficients (Equation 7) were examined using this approach. All but one failed to produce any relationship of interest. Consider again the modified equation of state known as the Helmbold Equation:

$$f(x, y, t) = \ln\left(\frac{x_0^2 - x^2}{y_0^2 - y^2}\right) = \alpha \ln\left(\frac{x_0}{y_0}\right) + \beta \quad (16)$$

The right hand side depends only on initial values and therefore is a constant. Hence $df = 0$. For this to hold true for all values of x, y and t , it must hold true separately for the component terms in both dt and dz after application of Equation 15 to the centre expression of Equation 16.

Considering the term for the dt component, an expression can be obtained which must be identically equal to zero in order that $df = 0$. After some rather lengthy algebra to simplify the expression, which the interested reader can find in Appendix A, the following relationship can be obtained that must also hold true in order that $df = 0$.

$$\frac{x^2}{y^2} \frac{a^2}{b^2} = \frac{\sigma_1^2}{\sigma_2^2} \frac{x_o^2}{y_o^2} \frac{\left(1 + \frac{x^2}{x_o^2}\right)}{\left(1 + \frac{y^2}{y_o^2}\right)} \quad (17)$$

The functional relationship between the strengths for both sides can be more easily seen if $X = x/x_o$ and $Y = y/y_o$. Then:

$$\frac{\left(1 + \frac{1}{Y^2}\right)}{\left(1 + \frac{1}{X^2}\right)} = F\left(\frac{x_o}{y_o}\right) \quad (18)$$

This relationship can be considered as a corollary to Helmbold's Equation and will be examined using the historical database in a subsequent section.

6. Analyses of the Distribution of Historical Data

A cursory examination of the analysis of the historical database presented in Figures 1 to 3 shows considerable scatter in the results around the mean values. Given the logarithmic scales employed, it is clear that the scatter in the results is as important for the relationship between dependent and independent variables as the underlying relationship between them of Equation 4. The pattern in the scatter of casualty results from the historical database has already been shown to be consistent with a log-normal distribution [7]. The observed log-

normal distribution of casualties was expected on the basis of an approximate solution to stochastic forms of Lanchester's Equations of Equation 7 and the observation that the historical data was predominantly from the region of the approximation's validity.

This section will examine the new expanded historical database for consistency with the results of the previous work and also expand the analysis of the patterns in the distributions in the results. The enlarged database now contains sufficient records to permit analysis of the distributions found from different segments of the overall database, and look for dependencies. It will also expand the examination to quantities other than those previously examined.

Ordinary Least Squares Regression (LSR) was used in this analysis in preference to the currently popular Maximum Likelihood Estimation (MLE) method [33]. MLE is believed to be more robust in handling data that do not meet the requirements for rigorous LSR (normally distributed residuals etc). However, with data known to be affected by bias, LSR is generally more useful in being easier to understand and because of the availability of well-established diagnostic tools [34]. The use of residual plots in particular was extensively used in the present work to identify data affected by bias and adjust the analysis accordingly.

6.1 Distribution of Initial Strengths

Prior to an examination of the distribution of battle casualties, it is necessary to consider the distribution of initial strength values for evidence of the bias referred to in Section 3.

The frequency distribution of initial force sizes was determined by dividing the range of force sizes into intervals of 10000 participants and counting the number of times a force strength from the database occurred in each interval. This is shown on a logarithmic scale in Figure 7. Noting the low correlation between the magnitude of attacker and defender force initial strengths (Figure 5), this distribution counts the initial strength of both sides separately rather than the combined forces total. Each battle therefore contributes two data points to the distribution.

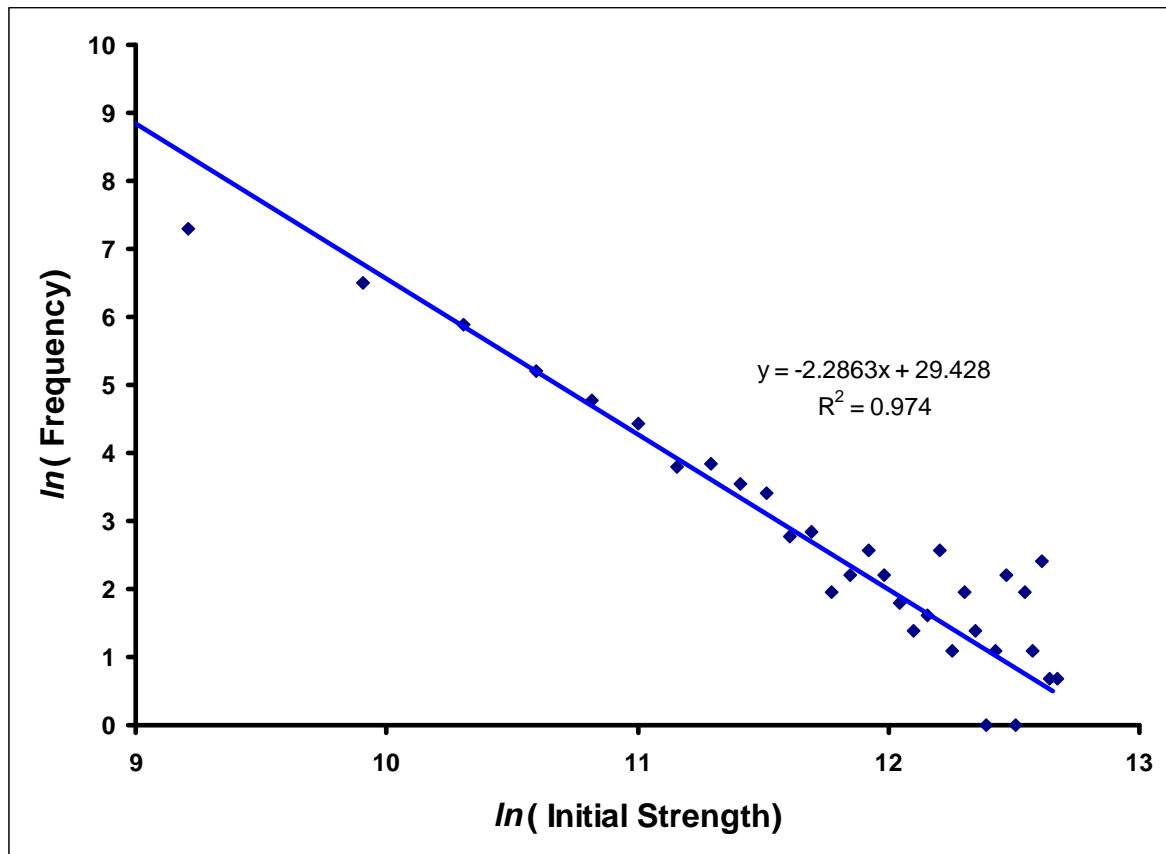


Figure 7: Force Size Distribution, and regression coefficient of determination

For the analysis presented in Figure 7, examination of the residuals confirmed the effect of small battle under-representation had biased the smallest data value. The difference between the observed value and expected value can be used to compensate for bias in the cumulative number distributions in the following analysis. The discrete nature of the dependent variable defines the measurement resolution. Where the value being measured is of the same size as the measurement resolution, the results can be strongly affected by fluctuations. Residual plots also identified which data is affected by fluctuations. In Figure 7 this corresponds roughly to the region where the logarithm of the Force Initial Strength is greater than 12, which could then be ignored by the regression analysis. The regression line (and coefficient of regression) shown in Figure 7 was produced after excluding those data points subject to bias and fluctuations.

It should also be noted that the distribution of initial force sizes does not exhibit any indication of the influence of a normal distribution. The completely different behaviour of the initial strength frequency and the casualty frequency supports the contention that such behaviour results from the attrition process and is not an artefact of the sampling or analysis procedure.

6.2 Distribution of Casualties

The distribution of the natural logarithm of each side's battle casualties was determined by dividing the range of observed logarithm of casualty values into intervals of size 1, which is equivalent to the size for adjacent intervals having a ratio of 1.65. It results in an even spread of casualty values on a logarithmic scale which is necessary for the accurate representation of its distribution. The number of times the logarithm of the casualty value from the database occurred in each interval was then counted. This is shown in Figure 8.

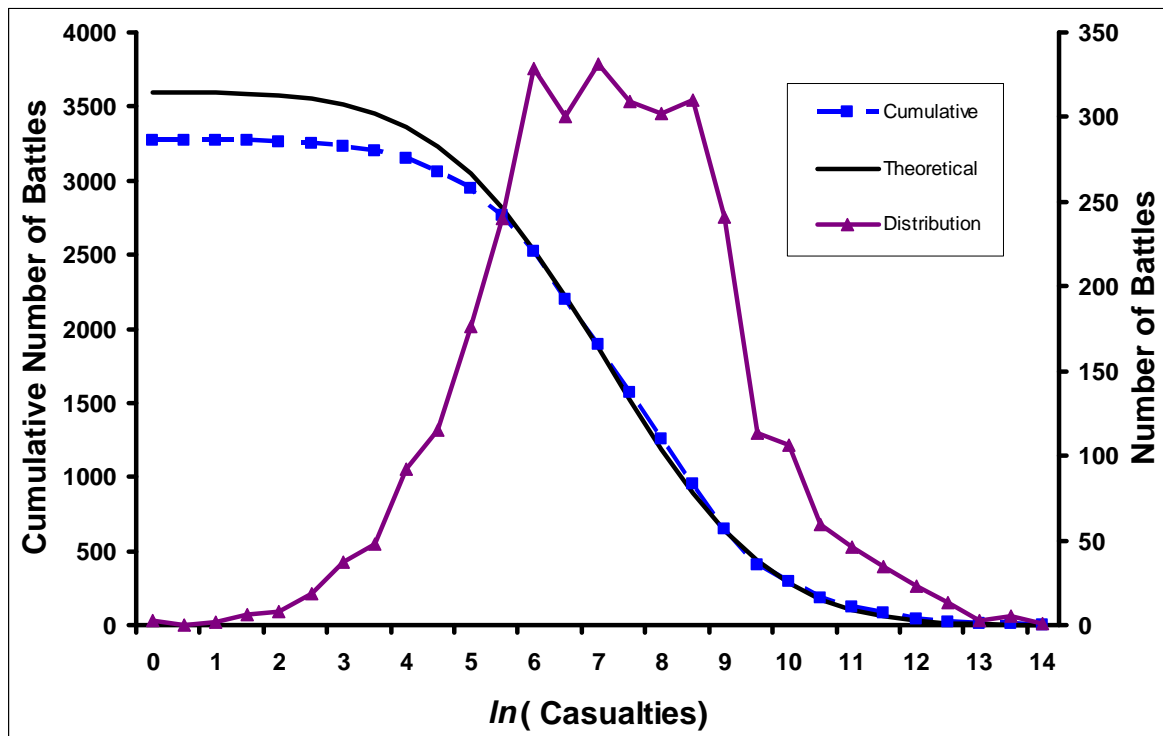


Figure 8: Number Distribution of Battles, Cumulative Distribution of Such Battles, and Theoretical Cumulative Normal Distribution vs. $\ln(\text{Casualties})$

The frequency distribution forms a bell shaped curve, but with considerable stochastic variability across the peak. This limits the ability to determine what form the distribution takes, in particular whether it is consistent with a normal distribution. The cumulative casualty distribution was formed by summing the number of occurrences with casualties greater than the specified value and is also shown in Figure 8. The cumulative distribution for occurrences greater than the specified value was chosen as it confines the effect of data bias to a few entries at the lower end of the scale instead of incorporating the bias in all the data points.

The previous section has established that a bias in favour of larger battles in the historical record does indeed exist. This bias can be allowed for by calculating the theoretical curve that the distribution would follow assuming a normal distribution with the mean and standard deviation of the historical data, but using the results from Figure 7 to estimate the number of small battles "missing" from the database. Figure 8 also shows this theoretical cumulative

frequency distribution. This analysis counts the casualties of both sides separately rather than the combined total. Each battle therefore contributes two data points to the distribution.

Ignoring the lowest strata of data, where bias is expected to produce under-representation, the close agreement between the historical data cumulative probability distribution and the expected theoretical probability distribution is apparent, as in the previous work. Ignoring the data points affected by bias, the correlation coefficient between the observed and theoretical distribution was evaluated as 0.997. These results are again consistent with the expectation of the proposed stochastic forms of Lanchester's Equations.

The database size used in the previous work and the prodigious amount of data required to undertake frequency analysis with any degree of reliability, limited that study to examination of the data as a single coherent sample. The larger size of the current database permits frequency analysis to be undertaken, with some degree of confidence, when the database is segmented according to the side's posture and size of the battle.

6.2.1 Segmented by Posture

The above analysis procedure was also applied to only the casualties of the attacking force from each battle in the database, the results of which are shown in Figure 9. The number of battles in the sample is of course half that obtained from Figure 7, as each battle previously contributed two casualty values (one attacker and one defender).

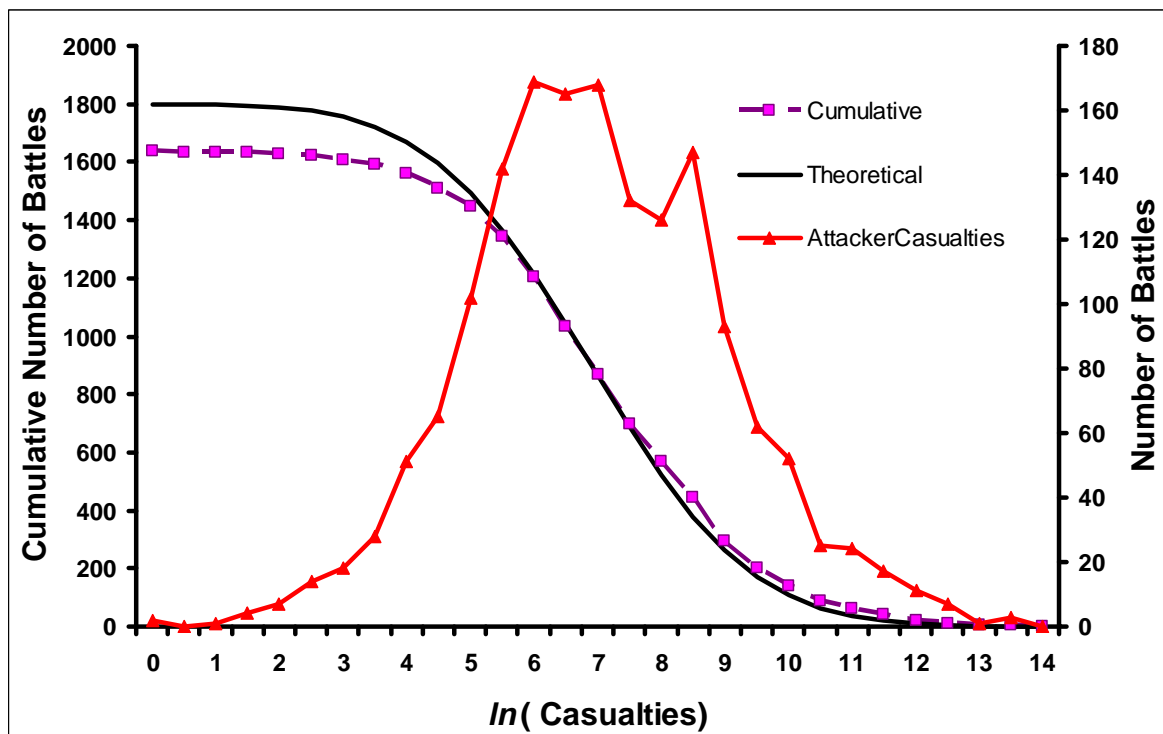


Figure 9: Number Distribution of Battles, Cumulative Distribution of Such Battles, and Theoretical Cumulative Normal Distribution vs. $\ln(\text{Casualties})$ for the attacking side.

A similar conclusion can be drawn regarding the behaviour of the casualty distribution for the attacking side, again consistent with the expectation of the stochastic forms of Lanchester's Equations. Repeating this procedure using only casualties for the defending side yields similar results.

6.2.2 Segmented by Outcome

A similar analysis can be undertaken when the data is segmented according to the battle's outcome and the casualty distributions of the winning and losing sides determined. Results very similar to those of Figure 9 were obtained for both sides and were consistent with the expectation of the stochastic forms of Lanchester's Equations. It is more interesting, however, to compare the observed distribution for the winning and the losing sides which are shown in Figure 10.

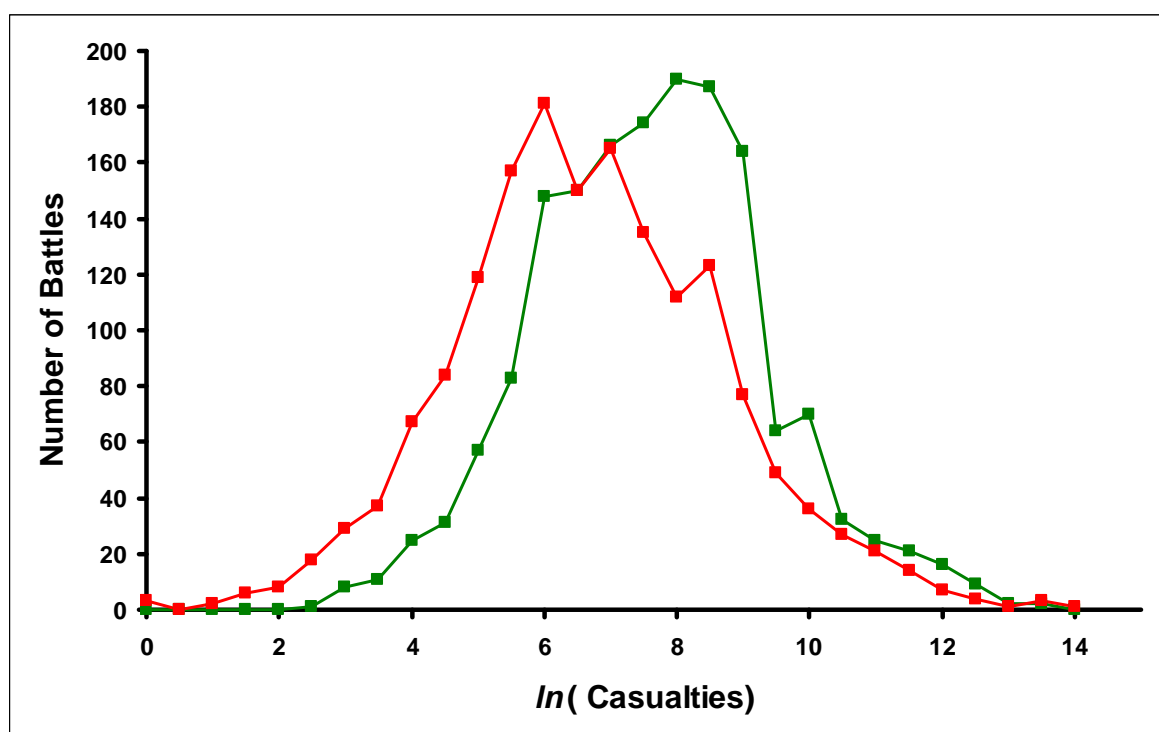


Figure 10: Number Distribution of Battles vs. $\ln(\text{Casualties})$. Winner casualties are coloured red while loser casualties are green.

The two casualty distributions are similar, but that for the loser has larger values for both the mean and variance than that for the winners. The difference in mean values is statistically significant. Examination of battle narratives may provide an explanation that does not imply greater rates of attrition. In particular, the inclusion of prisoners in the casualty values affects the loser more than the winner. While prisoners are an important component of casualties, especially in the determination of combat end points, they do not result from attrition. The losing side may also be subjected to pursuit and desertion, both of which affect the loser more than the winner. An effort was made during database construction not to include prisoners taken after combat in the reported casualty values, however prisoners taken during combat

cannot be easily separated from the casualties due to attrition in most of the data. All of which produces higher casualty values for the loser without the need to invoke a larger attrition rate.

While the difference between the mean values of the distributions are statistically significant, the difference between the variances is small enough that a 90% confidence limit cannot regard them as representing different distributions. Solutions to the stochastic differential equations [31] show that the mean value depends on both the systematic and stochastic contributions and hence on attrition rates, while the variance depends only on the stochastic part. These observations are consistent with stochastic processes acting evenly on both sides of a battle.

6.2.3 Segmented by Force Size

Examination of any dependence for the distribution of casualties on the size of the force was straightforward and just required the casualty values to be ordered according to its side's initial strength. Forces with the same initial strength but different casualty values were ordered according by the casualty value. The casualty values were then divided into quartiles, based on this ordering by force initial size. The use of quartiles to specify force size was necessary to ensure enough data was available to determine the casualty distribution with a degree of reliability. Summary statistics for the results are given below.

Table 4: Summary Statistics for $\ln(\text{Casualties})$ using Force Initial Size Quartiles

Quartile	Mean	Variance	Standard Error	Kurtosis	Skewness
1	5.28	2.31	0.05	-0.02	-0.39
2	6.38	1.69	0.04	-0.30	-0.31
3	7.05	1.70	0.05	-0.37	-0.25
4	8.95	2.26	0.05	0.19	0.18

The values for skewness and kurtosis are consistent with distributions close to normal, and hence log-normal for the casualty values. The change in sign for these quantities for the 4th Quartile (largest battles) may result from the inclusion of a small number of extremely large battles in the database. The resulting casualty distributions are shown in Figure 11. Further segmentation based on the side's posture was not possible, as can be seen from the variability across the peak of the each of the distributions.

Repeating the previous analysis for these results, indicates that the distributions are also consistent with the expectations of the stochastic attrition process. Mean casualty values increase with battle size (quartile) but the change in the variance is not statistically significant. The distribution variance depends on the magnitude of the stochastic contribution to the attrition rate in Equation 7 (σ_1 and σ_2) is again seen to act evenly in all battles.

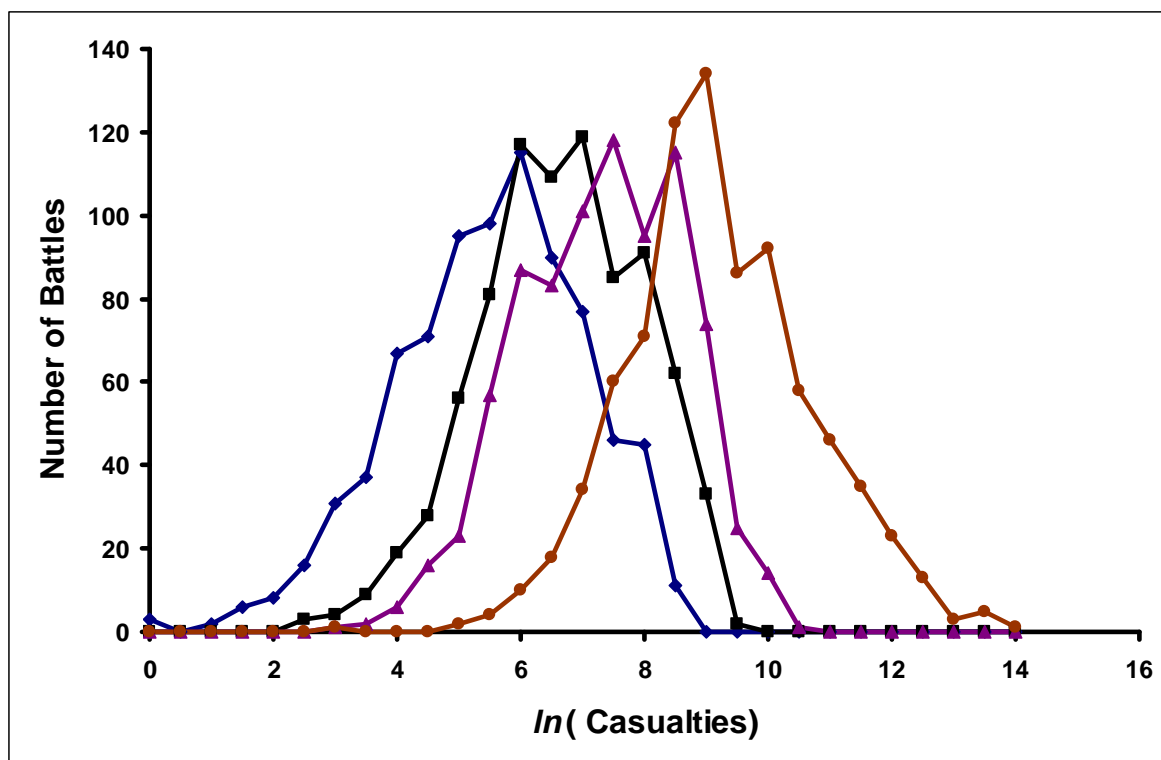


Figure 11: Number Distribution of Battles vs. $\ln(\text{Casualties})$ for different initial size forces. 1st Quartile values are blue, 2nd Quartile are black, 3rd Quartile are purple and 4th Quartile is brown.

6.3 Results Distributions in Helmbold's Relationship

Part of the motivation for the present work was to extend the examination of stochastic patterns in historical combat statistics beyond consideration of each side's casualty behaviour. Equation 14 shows that any arbitrary function defined from the system's stochastic variables should also exhibit stochastic behaviour. This should then be observable through examination of the frequency distribution of that function's value using historical data.

Section 5.2 examined the application of the differential rule of Equation 15 to the Helmbold Equation (Equation 4). However, analytic examination of the resulting expressions have yielded little of interest beyond the simple rule of Equation 18. Certainly, no clear indication of how the historical data (Figure 2) should be distributed about the mean value has been found.

The frequency distribution of the Helmbold Ratio in Figure 2 about the line of best fit can be obtained by modifying the procedure used in the previous section. Two different approaches for segmenting the database were examined. The posture of the winning side was used in the initial investigation, after which the effect of the size of the Force Ratio on the distribution was considered.

For each data point in Figure 2, the Force Ratio value was used to calculate an expected Helmbold Ratio using the line of best fit determined by regression analysis of the relevant

database segment. The value of the logarithm of the *historical* Helmbold Ratio minus the logarithm of the *expected* Helmbold Ratio is then calculated. The frequency distribution of these *delta-ln-Helmbold-Ratio* values is then determined using the procedure described in the previous section. Summary statistics for the number distributions using this database segmentation are given below with the distributions plotted in Figure 12.

Table 5: Summary Statistics by winner's postures of $\ln(\text{Helmbold Ratio Distribution})$

Posture	Mean	Variance	Standard Error	Kurtosis	Skewness
All	0.00	1.89	0.03	0.86	0.01
Attacker	0.00	1.25	0.03	0.72	-0.49
Defender	0.00	1.15	0.05	2.32	0.54

It is not clear whether the wide range of values for both the kurtosis and skewness is a result of the large fluctuations in the data or is indicative of underlying differences in the distributions. The analysis method is responsible for the mean values of zero. A normal distribution should have a kurtosis of 3 and a skewness of 0. The difference between the variances for attacker wins and defender wins is not statistically significant. The larger value for the variance observed for the entire database results from the offset between the regression lines for attacker wins and defender wins.

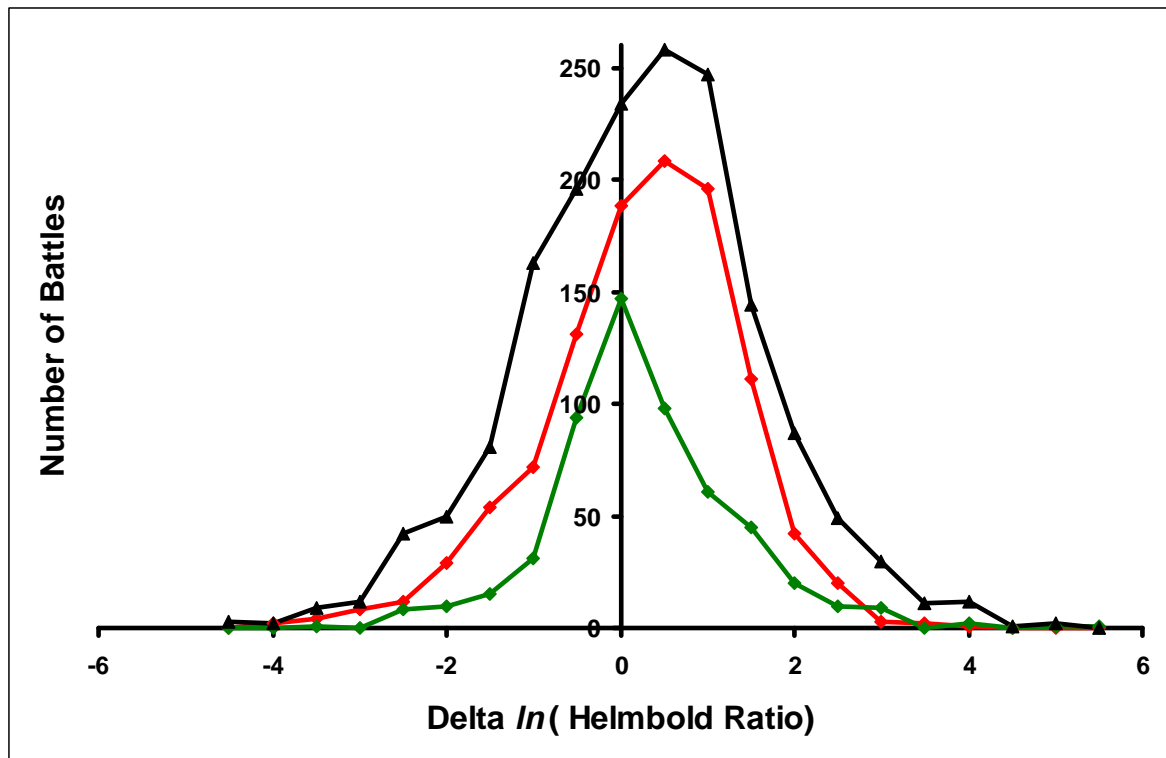


Figure 12: Helmbold Ratio number distribution. Attacker victories are coloured red while defender victories are green. Entire database is coloured black.

While this data does not allow the form of the distribution to be confirmed, the observed distributions are not inconsistent with the behaviour expected from a normal distribution.

6.3.1 Segmented by Battle Size

The size of the database limits the number of segments into which the data of Figure 2 can be split and still allow the distribution to be determined. Each data point in Figure 2 only contributes one value of the Helmbold Ratio, in contrast to two casualty values. Study of the dependence of the distribution of the Helmbold Ratio with battle size had to ignore the winner's posture. The number of battles is not sufficient to allow division into quartiles of sufficient size to facilitate frequency analysis with reasonable accuracy. The maximum number of size divisions that could be used was three. Even then, the fluctuations across the distribution was larger than desired. The dependence of the Helmbold Ratio on the Force Ratio suggests that the magnitude of the Force Ratio should be used to quantify battle size. Summary statistics for this segmentation are given in Table 6 and the distributions are plotted in Figure 13.

Table 6: Summary Statistics by battle size of $\ln(\text{Helmbold Ratio Distribution})$

Battle Size	\ln Force-Ratio		$\text{delta } \ln\text{-Helmbold-Ratio}$				
	Minimum	Maximum	Mean	Variance	Standard Error	Kurtosis	Skewness
Lower	-3.20	0.00	0.00	1.90	0.03	0.59	-0.09
Middle	0.01	0.71	0.03	1.73	0.03	1.96	0.02
Upper	0.72	3.66	-0.04	2.09	0.05	0.10	0.13

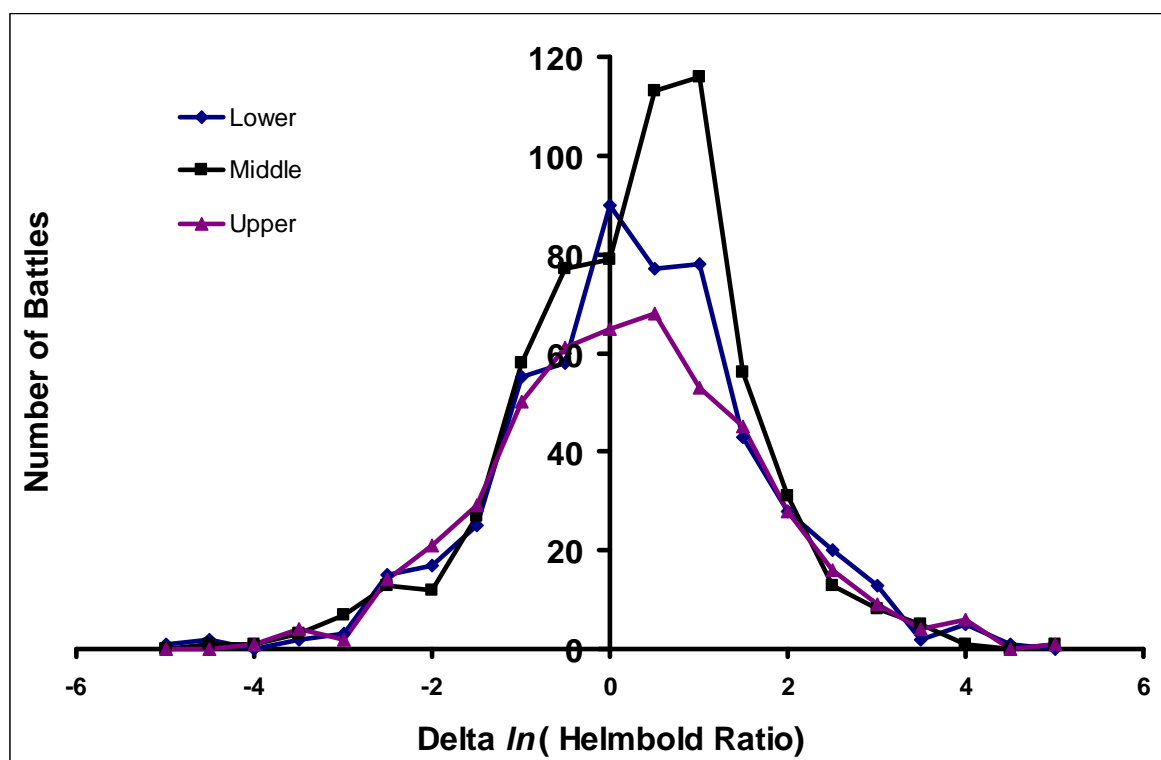


Figure 13: Helmbold Ratio number distribution dependence on Force Ratio magnitude. Lower third are coloured blue, middle third are coloured black and upper third are coloured purple.

The skewness and kurtosis show no definite trend with battle size as determined by the Force Ratio. The observed differences in the distributions are not statistically significant. The distribution of values for the Helmbold Ratio does not appear to depend on the Force Ratio in this analysis. This data does not allow the form of the distribution to be confirmed. However, the observed distributions are not inconsistent with the behaviour expected from a normal distribution.

7. Predictions of Battle Outcomes

Lanchester models of combat are generally considered to be capable of providing insight into a limited number of questions. The most important of which is 'What conditions are required for success?' (Who will win?) Helmbold [4] examined the correlation between the value of the Advantage parameter V and the battle's outcome:

$$V = \ln \mu = 2 \ln \left(\frac{1 - (x/x_0)^2}{1 - (y/y_0)^2} \right) \quad (19)$$

where x denotes the attacker and y the defender in each battle. The advantage parameter has been used in most subsequent work, including Hartley [5]. This simple relationship, using only initial and final strengths, does not include non-attrition considerations. It is a stochastic measure of success where a negative value predicts the attacker will be successful and a positive value predicts a defender's success. For this reason it is generally known as the Defender's Advantage V . To date, no studies to determine whether there is any dependence of the probability that V successfully predicts the outcome on other factors, such as the battle's size, appear to have been undertaken.

Examination of the effect of battle size on the probability that V successfully predicts the outcome has proven difficult. As already discussed, battle size is difficult to quantify as it can be measured in a number of ways. Most workers define the size of a battle as the total of all forces involved in the battle (both sides). The present work has found trends when battle size is determined by the strength of specific single sides. However, consideration of any dependence of the probability that V successfully predicts the outcome on battle size has been deferred to subsequent work.

The present work was only able to examine the historical database for any dependence on the winner's posture and battle date. The value of the Defender's Advantage and known battle outcome were used to determine the probability that the outcome was successfully predicted for each of the epochs (section 3.5) used in this database. This probability was determined for each epoch as a whole and also segmented by the posture of the winning side for each epoch. These results are listed in the following table. A date was assigned to the epoch by averaging the date for all battles constituting that sub-division.

Table 7: Probability that Defender's Advantage predicted outcome successfully

Dataset	Representative Year	Probability <i>Defender</i> successfully predicted	Probability <i>Attacker</i> successfully predicted	Probability <i>All</i> successfully predicted
Ancient	900	0.95	0.88	0.90
17 th Century	1650	0.67	0.94	0.85
18 th Century	1745	0.71	0.91	0.83
Revolution	1795	0.90	0.91	0.91
Empire	1810	0.74	0.83	0.80
ACW	1860	0.84	0.80	0.81
19 th Century	1865	0.67	0.85	0.78
WWI	1915	0.74	0.87	0.82
WWII	1945	0.56	0.81	0.74
Post WWII	1980	0.62	0.80	0.77

These results confirm that initial strengths and casualties are significant, if not dominant, determinants for successfully predicting battle outcomes. The results are also shown in Figure 14. It is interesting to note that the Defender's Advantage appears better at predicting overall attacker successes (0.85) than defender successes (0.73). This difference is statistically significant at the 95% confidence level.

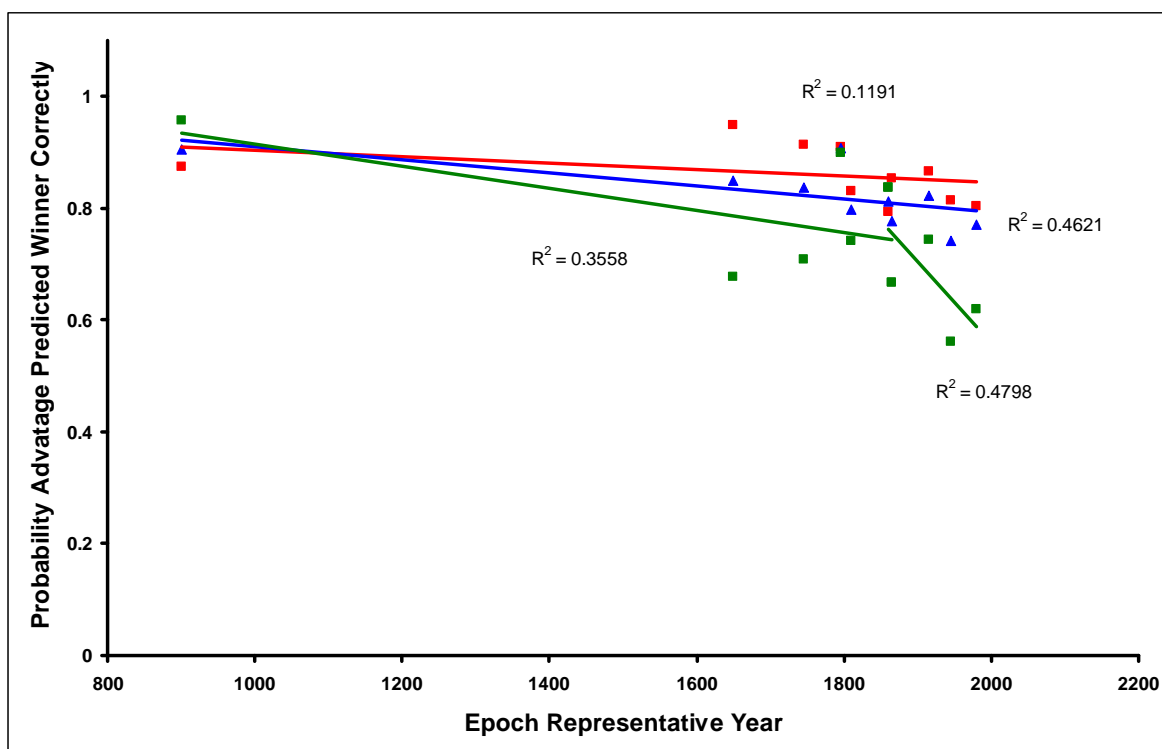


Figure 14: Probability that the Defender's Advantage correctly predicts the outcome segmented by epoch and posture. Attacker victories are coloured red while defender victories are green. Non-segmented data is coloured blue.

Figure 14 also includes straight line regression fits to the data. Little variation in the success rate of the advantage parameter with date, low values of the coefficient of determination, together with the lack of a systematic trend in the success rate of the advantage parameter is consistent with the independence of this parameter from date. This also implies that it does not depend on other quantities that correlate with date, such as the technology used in battle. There is one exception to this observation.

The behaviour of the probability of the advantage parameter to successfully predict the battle's outcome for all three cases considered is consistent up to around the year 1900. After that date the observed behaviour for attacker victories and for all battles remain consistent with each other and with their previous trend (consistent with no dependence on date). The behaviour for success in predicting defender victories, however, shows a noticeable change with the advantage parameter becoming much less successful at correctly predicting a defender victory. The change in the gradient of the lines of best fit between these two periods (pre-1900 and post-1900) is statistically significant, indicating that the observed reduction represents a real change in the contribution of attrition to the defender's success. At present no explanation for this behaviour has been found.

7.1 Combat Entropy

In the early 1990s the use of Shannon entropy (\mathbb{S}) to measure the disorder introduced as a result of combat attrition, where c is the number of casualties at time t and N the force strength at the same time was applied to combat [35]:

$$\mathbb{S} = -\frac{c(t)}{N(t)} \ln \frac{c(t)}{N(t)} \quad (20)$$

It was proposed that the difference in entropy between both sides would represent some measure of the outcome of the battle. This has subsequently been examined by a number of workers including Dexter [36], who found a good correlation between the entropy difference and battle outcome. The database used in that study was small (around 100 entries) and the criteria used for its construction are not clear, leaving unanswered questions about the impact of bias. Although better agreement was found using entropy than Lanchester predictions of victory (Defender's Advantage), the possibility of bias could not be excluded.

The ability of the entropy difference to correctly predict the outcome of a battle was also examined in the present work using the same methodology as for Defender's Advantage. The results also show a correlation between entropy difference and combat outcome, with the same trends as reported above using the Defender's Advantage parameter. For this reason, the results of the combat entropy study have not been reproduced. However, the present work found that combat-entropy was less effective in correctly predicting the outcome than the Defender's Advantage, in contrast to Dexter's conclusion [36].

7.2 Examination of the Corollary to Helmbold's Equation

The application of Ito's method to examine the differential of the Helmbold Ratio in section 5.2 produced the relationship of Equation 17, where the ratio on the left-hand-side (hereafter called the End Ratio) should depend only on the initial Force Ratio. The End Ratio depends only on the initial and final force values and has similarities to the force dependent parts of the Defenders Advantage parameter of Equation 19. All of which suggests that the End Ratio may also be of use to distinguish between battles won by the attacker from battles won by the defender.

The database developed for the present work can easily be used to examine this relationship, which is shown in Figure 15 using logarithmic scales along with the regression coefficient obtained from a least squares fit to each of the data segments. As usual, battles with attacker victories are coloured red while defender victories are coloured green.

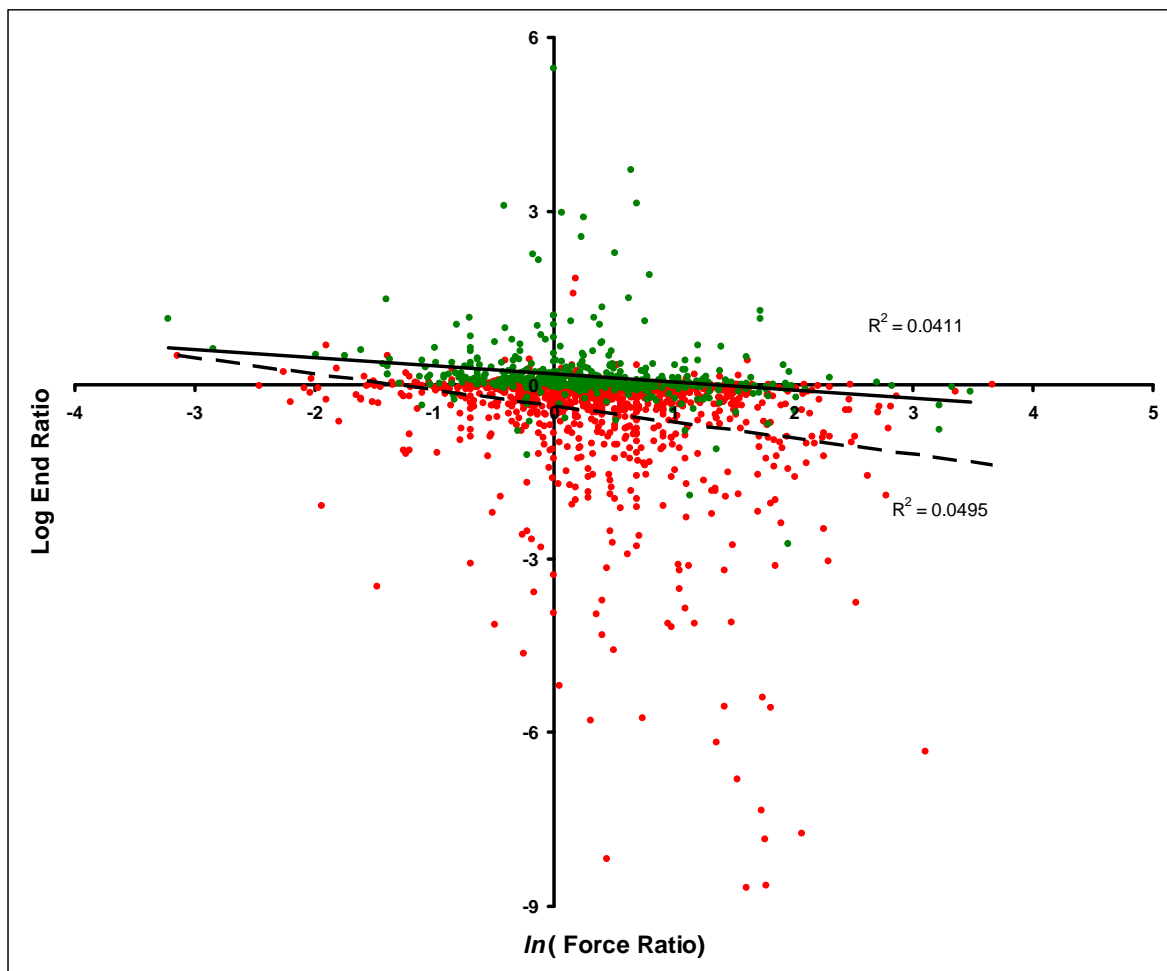


Figure 15: Relationship between the End Ratio and initial Force Ratio. Attacker victories are coloured red while defender victories are green.

Defender victories are seen to lie predominantly in the top half of Figure 15, with positive values for the logarithm of the End ratio, while attacker victories predominantly have

negative values for this parameter. The low values of the correlation coefficients obtained for each data segment, and shown in Figure 15, is consistent with no dependence of the End Ratio on the initial Force Ratio. The logarithm of the End Ratio does appear to differentiate between battles won by the attacker and battles won by the defender, with the value zero (corresponding to the End Ratio being unity) being the dividing line. The independence of the End Ratio from the Force Ratio is also consistent with the stochastic processes acting evenly on both sides of a battle.

8. Battle as a Complex Adaptive System

Lanchester's attrition equations describe the evolution of a single battle in time. However, analysis of historical data has shown that many of the relationships derived from Lanchester's Equations also describe the behaviour of the same parameters from such a collection of unrelated battles [4], [5], [6], [7]. Helmbold's pioneering work [4] made the assumption that the attrition coefficients were approximately the same for all battles. Hartley [5] sought to relax this assumption and has examined this issue at length. Neither has proposed a possible mechanism for why this behaviour is observed. Section 3.3 deferred consideration of that issue to here. Consideration of why this might be the case, albeit following an empirical rather than mathematically rigorous approach, requires a brief review of those Complex Adaptive Systems concepts necessary for an understanding of *scale free* behaviour.

The Lanchester Equations are a model for the behaviour of two interacting populations. Such dynamical systems (two interacting populations) are of considerable interest and have been widely studied. Another such system is the Lotka-Volterra model of predator-prey interacting populations, which has also been examined from a stochastic viewpoint [37]. This is of interest for the present work as it is widely believed to be an analogue for the Lanchester Equations.

The state of such systems is typically described by its location in phase space. The phase space coordinates in this case being restricted to the magnitudes of the populations, which are sufficient to describe the system's time dependence without the explicit inclusion of the rate of change as an additional coordinate. A defining characteristic of the Lotka-Volterra Equations is that their solutions encompass periodic behaviour. Put in other words, their phase space trajectories are closed curves, indicating that they describe conservative stable systems. (the same point in phase space can be revisited) This represents a significant difference to a system described by the Lanchester Equations. Such systems are fundamentally dissipative, their phase space trajectories have a point attractor with both force strengths equal to zero. The most useful properties of the Lotka-Volterra Equations therefore cannot be applied to the Lanchester Equations. This highlights the importance in understanding the general properties of dissipative systems to provide better insight into the dynamics of combat. Dissipative systems differ from conservative systems in a number of critical ways, the most important of which results from the reversible nature (at least in principle) of a conservative system compared with the fundamentally irreversible nature of a dissipative system.

Reversible systems are typically closed or isolated from external influence, their evolution is described by the current value of their endogenous variables whose past history is irrelevant.

They can be treated as if they are in equilibrium, or at least can be described as quasi-static systems, in which the system follows a succession of equilibrium states and transitions between the states sufficiently slowly such that at each moment of time the system can be treated as if it were in equilibrium. This model of system dynamics has been widely applied since the time of Newton, and with considerable success [38]. However, during the 20th century there has been an increasing acceptance that a whole class of problems has been largely ignored, because they are not suited to examination by the techniques developed in the study of reversible systems. Many dissipative systems are poorly described using techniques developed for reversible systems. Just as classical thermodynamics was used as a model for many reversible systems [38], the recent (and ongoing) development of irreversible thermodynamics [39] is beginning to be seen as a starting place for more appropriate studies of other irreversible systems [40].

Irreversible systems, in contrast, are typically open or interacting with their environment, their evolution also requires knowledge of appropriate exogenous variables. The development of stochastic forms of Lanchester's Equations being one approach to the inclusion of a model for that interaction into the theory of attrition. The past history of the system can be important for understanding its future development. If a system is reversible, its entropy does not change. Indeed, this can be used as a definition of a reversible process. Classical thermodynamics approximates real systems as the sum of closed quasi-static reversible and irreversible parts. The entropy of the irreversible part increases in accordance with the second law as the system evolves towards its equilibrium end-state, which consequently can be seen as the state with maximum entropy. This is a good description of the behaviour of closed systems. Many real systems are open in addition to being irreversible. The 1977 Nobel Prize for Chemistry was awarded in large part for demonstrating that such systems do not evolve to maximum entropy and equilibrium states [38]. Instead, they evolve to states for which the production of entropy is minimised, at least in the linear thermodynamics regime. This stationary state is maintained through its dynamic interaction with the environment, and allows the system to exist in a more structured condition than would be permitted at equilibrium.

Equilibrium is the condition of maximum entropy, which is also the state of minimum information content. The ensemble of components forming the system have their individual states distributed according to the Boltzmann distribution [40]. In the large number (size) limit, this can be replaced by the Poisson (discrete) or Gaussian (continuous) probability distributions resulting from the corresponding random processes [41].

These probability distributions exhibit a flat spectral density⁵ (white noise) for variations about their mean values, which is also regarded as characteristic of random processes.

$$S(\nu) = \text{constant} \quad (21)$$

This is the case for closed or isolated systems at equilibrium or quasi-static equilibrium. As demonstrated above, open or interacting systems are not subject to the same constraints as

⁵ White noise is a random signal and its frequency spectrum is determined from the Fourier Transform of its probability distribution. In this case a constant.

equilibrium systems and may consequently exhibit different spectral densities in their behaviour. This is usually interpreted as equivalent to meaning non-equilibrium systems are not described by random behaviour. While this is a sufficient condition, it is not a necessary condition.

The random processes underlying equilibrium behaviour all employ one key assumption, events occur independently of one another. When this does not hold, and there is partial correlation between events, the system contains additional information describing that correlation and white noise is no longer a description of the spectral density.

However, this behaviour can still arise from a random process. Consider a fishing fleet of N boats where the captain of each boat makes a random choice whether to fish on any given day independently of the decisions of the other captains. This leads to a simple Poisson (equilibrium) distribution for the expected number of boats observed fishing per day. If the effect of weather (an exogenous variable) on the expected return for a day's fishing is then added to this model, the probability that a boat will fish on a poor day will become less than the probability that it will fish on a good day. Each captain still makes the random decision to fish or not independently of the others, but the decisions are now correlated through the action of the weather in biasing the probability. This leads to a non-Poisson distribution of observed fishing boats and a spectral distribution that is frequency dependent, where the additional information content describes the pattern of the weather. This type of behaviour is commonly known as Self Organised Criticality.

Many such systems are known in nature and exhibit a characteristic frequency dependent spectral distribution commonly known as Pink noise [40, 42]:

$$S(\nu) \propto \frac{1}{\nu^s} \quad (22)$$

where the exponent is typically a real number $0 < s < 2$. This is equivalent to a frequency dependent probability for the occurrence of system events. This type of system is known as scale free, because regardless of the size (scale) of the spectrum under investigation the same behaviour is observed (a small piece of the spectrum examined in detail or a much larger section examined more coarsely). The same phenomena producing the microscopic behaviour is also responsible for the macroscopic behaviour. Scale free behaviour has been observed in many different dynamical systems, ranging from the frequency of earthquakes [30] to traffic accidents [43].

A number of descriptions for the emergence of self-organised criticality have been proposed, associated with the observation of pink noise. They are frequently used as analogues for other complex systems exhibiting scale free behaviour. In systems that are not static but evolving, the Red Queen Principle (or co-evolution) has also been suggested as a mechanism behind self-organised criticality [44]. The Red Queen Principle [45] is based on the observation to Alice by the Red Queen in Lewis Carroll's *"Through the Looking Glass"* that *"in this place it takes all the running you can do, to keep in the same place"*. If a number of dynamic systems coexist, the random variations introduced by evolution in one system can produce a competitive

advantage (increased fitness) for that system, and thus be able to capture a larger share of the resources available to all. This means that a fitness increase in one evolutionary system will tend to lead to a fitness decrease in another system. The only way that a system involved in a competition can maintain its fitness relative to the others is by in turn improving its design. In an evolutionary system, continuing development is needed in order to maintain its fitness relative to the systems it is co-evolving with.

A possible explanation of why the pattern of behaviour observed from an ensemble of different battles (Figure 1 and Figure 2) can be described using the behaviour expected within a battle (Equation 4) can now be attempted. This is an observation of scale free behaviour in which the same pattern of behaviour is observed at different scales of a phenomena. Stochastic forms of Lanchester's Equations include a model for the effect of the wider environment (exogenous variables) on attrition [7]. Exogenous variables imply that combat is an open system which allows non-equilibrium states to be stable. An evolving non-equilibrium system can result in the co-evolution of system variables and the Red Queen effect. Such variables will exhibit a power law (pink noise) relationship. The cause of scale free behaviour in attrition can then be understood as a consequence of the Red Queen effect. All that remains is to identify quantities involved in the attrition process that should exhibit co-evolution and do show a scale free (power law) relationship.

Dupuy [13] suggests an obvious co-evolution between weapon lethality and battlefield dispersion, although the data in that publication is not particularly suitable for quantitative study. The co-evolution can be explained using the following sequence of events. *Side 1* introduces (*evolution*) a more effective weapon or tactics, increasing the attrition coefficient of *side 2*. This changes the value of the values for the coefficients α and β in Equation 4 moving the battle data away from the line-of-best-fit. *Side 2* can choose to respond (*co-evolution*) to this change by spreading its forces out. This reduces the number of targets available to *side 1* and their rate of loss. The effective value of the attrition coefficient is also reduced which counters the previous changes in α and β , restoring the status-quo. The data from an ensemble of battles should then cluster about the line representing the evolution of a single battle with values of α and β describing an "average" battle.

Later work by Dupuy does include a limited amount of data to permit a preliminary investigation. Dispersion can be easily measured in terms of the number of troops per square kilometre. Lethality is more difficult to define, let alone measure. For an explanation of how lethality is defined and measured the interested reader is referred to Dupuy's work [46]. The dispersion and lethality results are shown on logarithmic axes in Figure 18 along with a least squares regression to the data.

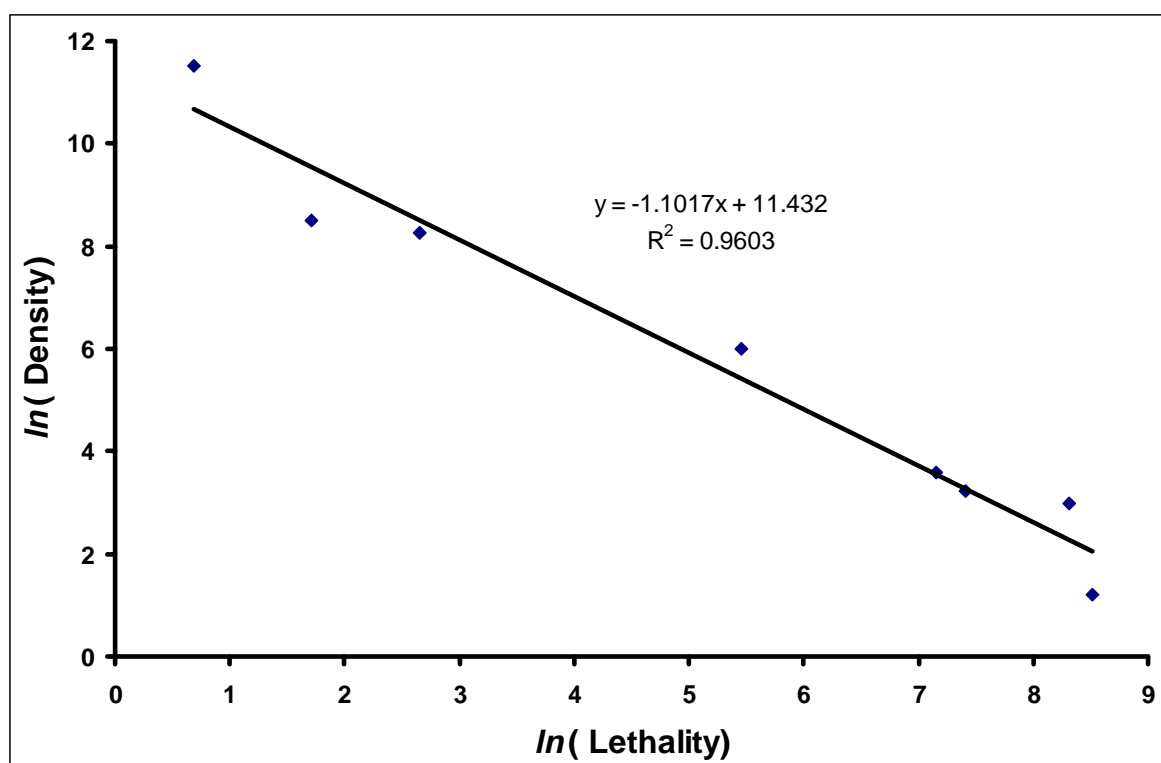


Figure 17: $\ln(\text{Troop Density})$ as a Function of $\ln(\text{Lethality})$

A power law relationship between lethality and dispersion is consistent with these results. This supports the contention that co-evolution is at work between quantities involved in attrition which acts to oppose any tendency for battle results to deviate consistently from the average. Scale free behaviour of some parameters should be expected, and is a possible explanation for the observed agreement in the behaviour of the historical ensemble of battles with the expectation of a single battle.

9. Conclusions

The present work has used two different approaches in its examination of the behaviour expected from a battle where attrition is described by Lanchester's Equations. It is important to remember that Lanchester's Equations are not a model of combat, only a model for combat attrition. The equations alone, therefore, cannot be expected to capture other effects such as the movement of engaged forces. Both methodologies can be regarded as empirical, rather than mathematically rigorous.

The present work, while not mathematically rigorous, has applied a standard stochastic calculus method to find the equation that specifies the evolution of an arbitrary function in a system with two stochastic variables. Similar to the approach of Black and Scholes, this differential form was applied to a number of functions known to be of interest for the Lanchester Square Law system of equations. Application of this to Helmbold's Ratio lead to

the discovery of a new construct, defined using both sides initial and final force strengths, that has proven able to differentiate between battles won the attacker and battles won by the defender.

The second approach, which constituted the bulk of the present work, compared historical battle data to the behaviour expected from a battle where attrition is described by Lanchester's Equations. This required a comprehensive examination of the issues, problems and constraints on using historical data for any form of analysis. It is an area often neglected by such studies.

All battle compilations are the product of the recursive application of a data sampling process. The population consists of all battles. This is first sampled to produce the set of all recorded battles. Many, especially smaller engagements, are never recorded. The requirement that both the initial and final values of forces strengths are known produces another sub-sampling stage to generate the set of all recorded battles with usable data. This sampling process also discriminates against smaller battles. Larger battles receive more attention and hence are more likely to have their attributes recorded. All battle databases are themselves samples of that sample. Even if the final sampling process was random, the process of recording history generates an intrinsic bias towards larger battles. This bias cannot be eliminated and any analysis technique must include procedures for identifying and dealing with that bias. A method for the identification of the effect of bias was examined. If analysis of the data gives the same results, both before and after the effects of bias in the data has been addressed, the result can be considered as insensitive to the effect of bias and indicative of actual behaviour in recorded history.

The present work, in considering a battle as the interaction between combat and non-combat quantities in a complex adaptive system has also shown how self-organised criticality can produce scale free behaviour (same pattern of behaviour at different scales of examination in the data). This is believed to be an explanation for the observation that the behaviour of the results from an ensemble of different battles can be described using that expected from the evolution of a single battle.

The results of this study of the behaviour of the initial and final strengths for both sides from an ensemble of historical battles is consistent with the expectations of systems using the stochastic Lanchester Equations. Segmentation of the data base according to a number of parameters including the force's initial strength and the posture of the winning side also yield results consistent with the stochastic Lanchester Equations. Importantly, the comparisons indicate that the stochastic parts of the attrition processes act evenly on both sides of the battle, regardless of how the database was segmented.

The missing element of combat theory that rigorously links all the studies in the present work together is a theory of combat termination. No general or accepted theory for combat termination has to date been developed that agrees with the available historical results. This is clearly the major remaining problem in the development of a quantitative model of combat.

10. References

1. *Aircraft in Warfare: The Dawn of the Fourth Arm*, F W Lanchester, Constable & Co., London, (1916).
2. *Models, Data and War: A Critique of the Foundation for Defense Analyses*, US General Accounting Office, Report No. PAD 80-21, (1980)
3. *The Calculus of Conventional War: Dynamic Analysis without Lanchester*, *Studies in Defence Policy Series*, J M Epstein, The Brookings Institution, Washington DC, (1985).
4. *Historical Data and Lanchester's Theory of Combat Pt. 1*, R L Helmbold, CORG-SP-128, (1961).
5. *Topics in Operations Research: Predicting Combat Effects*, D S Hartley III, INFORMS, Linthicum Md, (2001).
6. *Art of war hidden in Kolmogorov's equations*, M K Lauren, G C McIntosh, N D Perry and J Moffat, *Chaos* **17**, (2007).
7. *Application of Black Scholes Complexity Concepts to Combat Modelling*, N D Perry, Australian Department of Defence, DSTO-TR-2318, (2009).
8. *Mathematical Methods in Defense Analyses*, 3rd Ed, J S Przemieniecki, AIAA Inc, Reston VA, (2000).
9. *Artificial War: Multiagent-Based Simulation of Combat*, A Ilachinski, World Scientific, Singapore, (2004).
10. *The Blind Watchmaker*, R Dawkins, Norton & co., New York, (1986).
11. *Lanchester Models of Warfare Volumes 1 and 2*, J G Taylor, Operations Research Society of America, (1983).
12. *De Physica Belli: An introduction to Lanchestrian Attrition Mechanics Part I*, B W Fowler, DMSTTIAC SOAR 95-01, (1995).
13. *Numbers Predictions and War*, T Dupuy, Macdonald and Janes, London, (1979).
14. *More is Different*, P W Anderson, *Science* **177** 393-396, (1972).
15. *Reinventing Medicine: Beyond Mind Body to a new era of Healing*, L Dosey, Harper SF (1999).
16. *Military Operations Research: Quantitative Decision Making*, N K Jaiswal, Kluwer Boston Ma., (1997).
17. *A Verification of Lanchester's Law*, J H Engel, *Operations Research* **2**, 163-171, (1954).
18. *An attempt to verify Lanchester's Equations*, J Busse, *Developments in Operations Research* **2**, 587-597, Gordon and Breach, New York, (1971).
19. *Lanchester Models of the Ardennes Campaign*, J Bracken, *Warfare Modelling*, 417-435, MORS Alexandria Va, (1995).
20. *The Effect of Battle Circumstances on Fitting Lanchester Equations to the Battle of Kursk*, T Lucas and J Dinges, *MORS* **9**, 17-30, (2004).
21. *On Sample Size and Precision in Ordinary Least Squares*, A Montenegro, *J. Appl. Stat.* **28**, 603-605, 2001.
22. *A manual of sampling techniques*, R K Som, Heinemann, London, (1973).
23. *Bootstrap Methods for Standard Errors, Confidence Intervals, and Other Measures of Statistical Accuracy*, B. Efron and R. Tibshirani, *Statistical Science* **1**, 54-77, (1986).
24. http://en.wikipedia.org/wiki/List_of_battles (accessed August 2009).
25. *The Greenhill Napoleonic Wars Data Book*. D Smith, Greenhill Books, London, (1998).

26. *Wisdom of crowds: why the many are smarter than the few and how collective wisdom shapes business, economies, societies and nations*, J Surowiecki, Random House, London, (2004).
27. *The Military Landscape*, J T Dockery and A E R Woodcock, Woodhead Press, Camb. UK, (1993).
28. *Instabilities and catastrophes in science and engineering*, J M T Thompson, Wiley, Chichester UK, (1982).
29. *Stochastic Attrition Models of Lanchester Type*, A F Karr, report no. P-1030, Institute for Defense Analyses, Arlington VA, 1974.
30. *Introduction to stochastic calculus with applications*, F Klebaner, Imperial College Press, London, 1998.
31. *Stochastic versions of Lanchester equations in wargaming*, M Amacher and D Mandallaz, Eur. J. Oper. Res. **24**, 41-45, (1986).
32. *The Pricing of Options and Corporate Liabilities*, F Black and M Scholes, J. Pol. Econ. **81**, 673-659, 1973.
33. *Power law distributions in empirical data*, A Clauset, C R Shalizi and M E J Newman, SIAM Review, In press, (2009).
34. *The use of Partial Residual Plots in Regression Analysis*, W A Larson and S J McCleary, Technometrics, **13** 781-790, (1972).
35. *A proposed Entropy Measure for Assessing Combat Degradation*, F Carvalho-Rodrigues, J. Oper. Res. Soc. **40**, 789-793, (1989).
36. *Combat entropy as a measure of effectiveness*, P Dexter, J Battlefield Technology **6**, 33-39, (2003).
37. *Long term behaviour of solutions of the Lotka-Volterra system under small random perturbations*, R Z Khasminskii and F C Klebaner, The Ann. of Appl. Prob. **11**, 952-963, (2001).
38. *The Origin of Wealth*, E Beinhocker, Harvard Business School Press, Boston Mass., (2007).
39. *Thermodynamic theory of structure, stability and fluctuations*, P Glansdorff and I Prigogine, Wiley Interscience London, (1971).
40. *Deep Simplicity*, J Gribbin, Penguin, London, (2005).
41. *Elements of statistical thermodynamics*, L K Nash, Addison Wesley Reading Mass., (1971).
42. *Linked*, A L Barabasi, Perseus Press, Camb. Mass, USA, (2002).
43. *Some statistical aspects of road safety research*, R. J. Smeed., Journal of the Royal Statistical Society. Series A (General) **112**, (1949).
44. *Self organised criticality in living systems*, C Adami, Phys. Lett. A **203**, 29-32, (1995).
45. *A New Evolutionary Law*, L Van Valen, Evolutionary Theory **1**, 1-30, (1973).
46. *The Evolution of Weapons and Warfare*, T Dupuy, Bobbs-Merrill, New York, (1980).

Appendix A: Application of Ito's Rule to Helmbold's Relationship

Let:

$$f(x, y, t) = \ln\left(\frac{x_0^2 - x^2}{y_0^2 - y^2}\right) = \alpha \ln\left(\frac{x_0}{y_0}\right) + \beta \quad (\text{A1})$$

Given that:

$$df = \left(\frac{\partial f}{\partial t} - ay \frac{\partial f}{\partial x} dx - bx \frac{\partial f}{\partial y} dy + \frac{1}{2} \sigma_1^2 y^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} \sigma_2^2 x^2 \frac{\partial^2 f}{\partial y^2} + \sigma_1 \sigma_2 \rho xy \frac{\partial^2 f}{\partial x \partial y} \right) dt + \left(\sigma_1 y \frac{\partial f}{\partial x} + \sigma_2 \rho x \frac{\partial f}{\partial y} \right) dz \quad (\text{A2})$$

The term in the first bracket (the dt term) can be found by substituting the differentials:

$$2xy \left(\frac{b}{y^2 - y_o^2} - \frac{a}{x^2 - x_o^2} \right) - \sigma_1^2 y^2 \frac{(x^2 + x_o^2)}{(x^2 - x_o^2)} + \sigma_2^2 x^2 \frac{(y^2 + y_o^2)}{(y^2 - y_o^2)} \quad (\text{A3})$$

The first bracket in Equation A3 can be re-expressed as:

$$\frac{1}{x^2 - x_o^2} \left(b \frac{x^2 - x_o^2}{y^2 - y_o^2} - a \right) \quad (\text{A4})$$

However, from the equation of state (main text Equation 2) which is consistent with the modified form of the equation of state in Equation A1 (see the discussion prior to Equation 4 in the main text) shows that:

$$\frac{x^2 - x_o^2}{y^2 - y_o^2} = \frac{a}{b} \quad (\text{A5})$$

Hence the first bracket of Equation A3 is zero. Therefore:

$$\sigma_2^2 x^2 \frac{(y^2 + y_o^2)}{(y^2 - y_o^2)} - \sigma_1^2 y^2 \frac{(x^2 + x_o^2)}{(x^2 - x_o^2)} = 0 \quad (\text{A6})$$

Equation A5 can also be used to replace the $x^2 - x_o^2$ term in Equation A6, leaving:

$$\sigma_2^2 x^2 (y^2 + y_o^2) = \sigma_1^2 y^2 \frac{b^2}{a^2} (x^2 + x_o^2) \quad (\text{A7})$$

From which some elementary cross-multiplication produces the desired result: of Equation 17 from the main text.

DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION DOCUMENT CONTROL DATA					
				1. PRIVACY MARKING/CAVEAT (OF DOCUMENT)	
2. TITLE Applications of Historical Analyses in Combat Modelling			3. SECURITY CLASSIFICATION (FOR UNCLASSIFIED REPORTS THAT ARE LIMITED RELEASE USE (L) NEXT TO DOCUMENT CLASSIFICATION) <div style="display: flex; justify-content: space-between;"> Document (U) </div> <div style="display: flex; justify-content: space-between;"> Title (U) </div> <div style="display: flex; justify-content: space-between;"> Abstract (U) </div>		
4. AUTHOR(S) Nigel Perry			5. CORPORATE AUTHOR DSTO Defence Science and Technology Organisation Fairbairn Business Park Department of Defence Canberra ACT 2600 Australia		
6a. DSTO NUMBER DSTO-TR-2643		6b. AR NUMBER AR-015-190		6c. TYPE OF REPORT Technical Report	
7. DOCUMENT DATE December 2011					
8. FILE NUMBER 2011/1103392		9. TASK NUMBER 07/279		10. TASK SPONSOR RL CAP	
				11. NO. OF PAGES 48	
				12. NO. OF REFERENCES 46	
13. DSTO Repository of Publications http://dspace.dsto.defence.gov.au/dspace/			14. RELEASE AUTHORITY Chief, Joint Operations Division		
15. SECONDARY RELEASE STATEMENT OF THIS DOCUMENT <div style="text-align: center;"><i>Approved for public release</i></div>					
OVERSEAS ENQUIRIES OUTSIDE STATED LIMITATIONS SHOULD BE REFERRED THROUGH DOCUMENT EXCHANGE, PO BOX 1500, EDINBURGH, SA 5111					
16. DELIBERATE ANNOUNCEMENT No Limitations					
17. CITATION IN OTHER DOCUMENTS Yes					
18. DSTO RESEARCH LIBRARY THESAURUS http://web-vic.dsto.defence.gov.au/workareas/library/resources/dsto_thesaurus.shtml Combat models Probabilistic Modelling Historical Analysis Complex Adaptive Systems					
19. ABSTRACT While Lanchester's equations are commonly used as the basis for force-on-force combat models, it is important to remember that Lanchester's Equations are not a model of combat, only a model for combat attrition. There have been numerous attempts to compare historical combat data with the behaviour expected from Lanchester's Equations.					